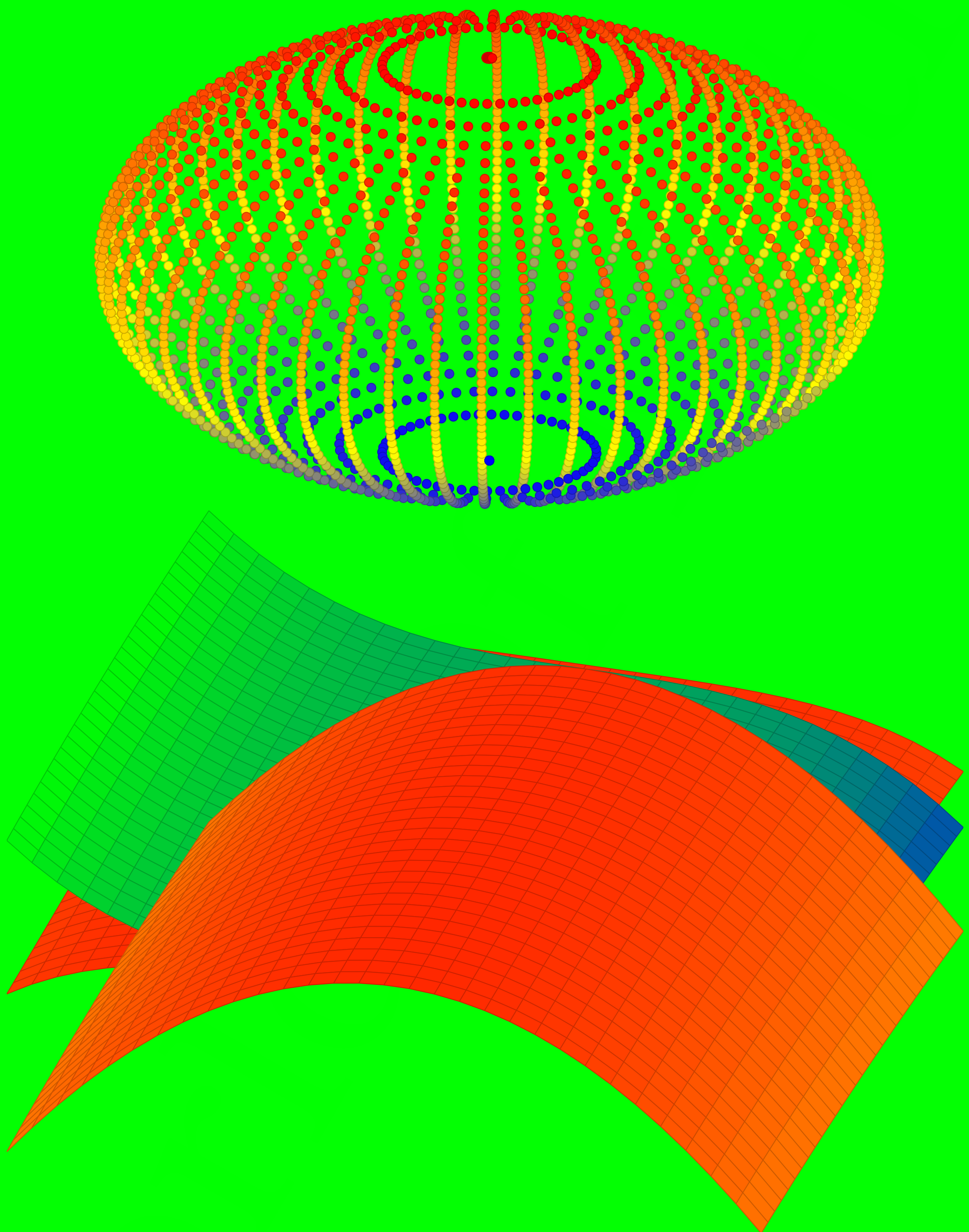
SOLUTIONS TO POLYNOMIAL EQUATIONS 1

BY USING SUBSTITUTIONS



★LINEAR★ ★QUADRATIC★ ★CUBIC★

♦ BY FARAI SIKITENI ♦

WORLD OF MATHEMATICS NOTES

SOLUTIONS TO POLYNOMIAL EQUATION 1

BY FARAI SIKITENI

November 26, 2020

Contents

1 INTRODUCTION

A polynomial equation is an equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

where $a_0 \neq 0$ *and* $n \in \mathbb{N}$. n is the degree of a polynomial equation.
In this book , we are going to discuss how to solve linear , quadratic and cubic equa-

Degree	Name	Form ($a \neq 0$)
1	Linear equation	$ax + b = 0$
2	Quadratic equation	$ax^2 + bx + c = 0$
3	Cubic equation	$ax^3 + bx^2 + cx + d = 0$
4	Quartic equation	$ax^4 + bx^3 + cx^2 + dx + e = 0$
5	Quintic equation	$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

tions by radicals.
A radical(analytic) solution is a basic solution of x *,that is,* x *is expressed in terms of the coefficients of a polynomial equation.*

Assumed knowledge

- you know how to solve linear equations
- you know how to solve quadratic equations using the method of completing the square or by factorisation

So we are not going to discuss these simple methods but we are going to use longer methods (using substitutions) to solve linear and quadratic equation so that it will be easy for you to understand how to solve the cubic equations.

2 LINEAR EQUATIONS

2.1 Solution to Linear equations

Solve the equation $ax + b = 0$ where $a \neq 0$

Solution

★$ax + b = 0$

Substitution 1

let $x = y - \frac{b}{a}$

$\Rightarrow a(y - \frac{b}{a}) + b = 0$

$\Rightarrow a(\frac{ay-b}{a}) + b = 0$

$\Rightarrow (ay - b) + b = 0$

$\Rightarrow ay - b + b = 0$

$\Rightarrow ay = 0$

$\Rightarrow y = 0$

Back substitution for sub 1

$\Rightarrow x = y - \frac{b}{a} = 0 - \frac{b}{a}$

$\Rightarrow \underline{\underline{x = -\frac{b}{a}}}$

2.1.1 Example 1

Solve the equation $3x - 4 = 0$

Solution

$3x - 4 = 0$

Substitution 1

Let $x = y - \frac{-4}{3} = y + \frac{4}{3}$

$\Rightarrow 3(y + \frac{4}{3}) - 4 = 0$

$\Rightarrow 3(\frac{3y+4}{3}) - 4 = 0$

$\Rightarrow 3y + 4 - 4 = 0$

$\Rightarrow 3y = 0$

$\Rightarrow y = 0$

Back substitution for sub 1

$\Rightarrow x = y + \frac{4}{3} = 0 + \frac{4}{3}$

$\Rightarrow \underline{\underline{x = \frac{4}{3}}}$

2.1.2 Example 2

Solve the equation $8x + 4 = 4x + 3$

Solution

$8x + 4 = 4x + 3$

Write the equation in the form $ax + b = 0$

$\Rightarrow 4x + 1 = 0$

Substitution 1

Let $x = y - \frac{1}{4}$

$\Rightarrow 4(y - \frac{1}{4}) + 1 = 0$

$\Rightarrow 4(\frac{4y-1}{4}) + 1 = 0$

$\Rightarrow 4y - 1 + 1 = 0$

$\Rightarrow 4y = 0$

$\Rightarrow y = 0$

Back substitution for sub 1

$\Rightarrow x = y - \frac{1}{4} = 0 - \frac{1}{4}$

$\Rightarrow \underline{\underline{x = -\frac{1}{4}}}$

3 QUADRATIC EQUATIONS

3.1 Solution to quadratic equations

Solve the equation $ax^2 + bx + c = 0$ where $a \neq 0$

Solution

Firstly , we must eliminate a term in x (i.e to depress a quadratic equation) by using the substitution $x = y - \frac{b}{2a}$

Substitution 1

Let $x = y - \frac{b}{2a}$

$\Rightarrow a(y - \frac{b}{2a})^2 + b(y - \frac{b}{2a}) + c = 0$

$\Rightarrow a(\frac{2ay-b}{2a})^2$+$b(\frac{2ay-b}{2a})$+c=0

$\Rightarrow a(\frac{4a^2y^2-4aby+b^2}{4a^2})$+$b(\frac{2ay-b}{2a}) + c = 0$

$\Rightarrow ay^2 - by + \frac{b^2}{4a} + by - \frac{b^2}{2a} + c = 0$

$\Rightarrow ay^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c = 0$

$\Rightarrow ay^2 + \frac{b^2-2b^2+4ac}{4a} = 0$

$\Rightarrow ay^2 - \frac{b^2-4ac}{4a} = 0$

NB∼ The equation above is a depressed quadratic equation ($a\prime y^2 + d\prime = 0$ *)*

Substitution 1

Let $w = y^2$

$\Rightarrow aw - \frac{b^2-4ac}{4a} = 0$

NB $\sim$ the equation in w is a linear equation ($a\prime w + b\prime = 0$) and the solution for linear equations is $w = -\frac{b\prime}{a\prime}$

$\Rightarrow w = -\frac{-\frac{b^2-4ac}{4a}}{a}$

$\Rightarrow w = - - \frac{b^2-4ac}{4a^2}$

$\Rightarrow w = \frac{b^2-4ac}{4a^2}$

<u>Back Substitutions</u>

<u>back substitution for sub 2</u>

$\bigstar w = y^2$

$\Rightarrow y = \sqrt{w}$

<u>back substitution for sub 1</u>

$\Rightarrow x = y - \frac{b}{2a}$

$\Rightarrow \underline{\underline{x = \sqrt{w} - \frac{b}{2a} \text{ where } w = \frac{b^2-4ac}{4a^2}}}$

<u>When</u> <u>$w \geq 0$</u>

$\Rightarrow x = \sqrt{w} - \frac{b}{2a}$

$\Rightarrow x = \sqrt{\frac{b^2-4ac}{4a^2}} - \frac{b}{2a}$

$\Rightarrow x = \pm\frac{\sqrt{b^2-4ac}}{2a} - \frac{b}{2a}$

$\Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

$\Rightarrow \underline{\underline{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}}$ *Quadratic Formula*

3.1.1 Example 1

Solve the equation $2x^2 - 5x + 3 = 0$

Solution

$\bigstar 2x^2 - 5x + 3 = 0$

<u>Substitution 1</u>

Let $x = y - \frac{b}{2a} = y + \frac{5}{4} = \frac{4y+5}{4}$

$\Rightarrow 2(\frac{4y+5}{4})^2 - 5(\frac{4y+5}{4}) + 3 = 0$

$\Rightarrow 2(\frac{16y^2+40y+25}{16}) - 5(\frac{4y+5}{4}) + 3 = 0$

$\Rightarrow (\frac{32y^2+80y+50}{16}) - (\frac{20y+25}{4}) + 3 = 0$

$\Rightarrow 32y^2 + 80y + 50 - 80y - 100 + 48 = 0$

$\Rightarrow 32y^2 - 2 = 0$

Substitution 2

Let $w = y^2$

$\Rightarrow 32w - 2 = 0$

Solving the linear equation in w

$\Rightarrow w = -\frac{-2}{32}$

$\Rightarrow w = \frac{1}{16}$

Back Substitutions

back substitution for sub 2

$\bigstar w = y^2$

$\Rightarrow y = \pm\sqrt{w} = \pm\sqrt{\frac{1}{16}} = \pm\frac{1}{4}$

back substitution for sub 1

$\Rightarrow x = \frac{4y+5}{4} = \frac{4(\pm\frac{1}{4})+5}{4}$

$\Rightarrow \underline{\underline{x = 1 \text{ or } x = \frac{3}{2}}}$

3.1.2 EXERCISE 1

Solve the following equations using suitable substitutions

1. $4x + 6 = 0$
2. $3x + 5 = 7$
3. $2x - 5 = 8x - 9$
4. $19x - 15 = 25x - 33$
5. $20x - 23 = 23$
6. $x^2 - 7x + 12 = 0$
7. $x^2 + 5x - 50 = 0$
8. $3x^2 - 2x = 5$
9. $9x^2 - 20 = 5$
10. $5x^2 + 25 = 0$
11. $x^2 + x + 1 = 0$
12. $x^2 - 5x + (7 + i) = 0$
13. $x^2 + (a + b)x + ab = 0$ where a and b are constants
14. $x^2 - 2c^2x + (c^4 - f^2) = 0$ where c and f are constants

4 CUBIC EQUATIONS

4.1 Solution to cubic equations

Solve the equation $ax^3 + bx^2 + cx + d = 0$ where $a \neq 0$

Solution

$\bigstar ax^3 + bx^2 + cx + d = 0$

Depress the cubic equation (eliminate the term in x^2) ,by making the substitution $x = y - \frac{b}{3a}$.

Substitution 1

Let $x = y - \frac{b}{3a}$

$\Rightarrow a(y - \frac{b}{3a})^3 + b(y - \frac{b}{3a})^2 + c(y - \frac{b}{3a}) + d = 0$

$\Rightarrow a(\frac{3ay-b}{3a})^3 + b(\frac{3ay-b}{3a})^2 + c(\frac{3ay-b}{3a}) + d = 0$

$\Rightarrow a(\frac{27a^3y^3-27a^2by^2+9ab^2y-b^3}{27a^3}) + b(\frac{9a^2y^2-6aby+b^2}{9a^2}) + c(\frac{3ay-b}{3a}) + d = 0$

multiply each term by $27a^2$

$\Rightarrow 27a^3y^3 - 27a^2by^2 + 9ab^2y - b^3 + 27a^2by^2 - 18ab^2y + 3b^3 + 27a^2cy - 9abc + 27a^2d = 0$

$\Rightarrow 27a^3y^3 + (9ab^2 - 18ab^2 + 27a^2c)y + (-b^3 + 3b^3 - 9abc + 27a^2d) = 0$

devide each term by $27a^3$

$\Rightarrow y^3 + (\frac{3ac-b^2}{3a^2})y + (\frac{2b^3+27a^2d-9abc}{27a^3})$

Let $e = \frac{3ac-b^2}{3a^2}$ *and* $f = \frac{2b^3+27a^2d-9abc}{27a^3}$

NB $\sim$ these are not substitutions since e and f are constants ie they are independent of y

$\Rightarrow y^3 + ey + f = 0$

NB $\sim$ This cubic equation is called a depressed cubic equation (a cubic equation that has no term in y^2 (x^2 if the unknown is x)). So if you are given a depressed cubic equation , there is no need for substititution $x = y - \frac{b}{3a}$. This implies that the substitution $x = y - \frac{b}{3a}$ is made to depress a cubic equation.

Substitution 2

Let $y = z + \frac{s}{z}$ *where* s *is a constant*

$\Rightarrow (z + \frac{s}{z})^3 + e(z + \frac{s}{z}) + f = 0$

$\Rightarrow (\frac{z^2+s}{z})^3 + e(\frac{z^2+s}{z}) + f = 0$

$\Rightarrow (\frac{z^6+3sz^4+3s^2z^2+s^3}{z^3}) + (\frac{ez^2+es}{z}) + f = 0$

multiply each term by z^3

$\Rightarrow z^6 + 3sz^4 + 3s^2z^2 + s^3 + ez^4 + esz^2 + fz^3 = 0$

$\Rightarrow z^6 + 3sz^4 + ez^4 + fz^3 + 3s^2z^2 + esz^2 + s^3 = 0$

$\Rightarrow z^6 + (3s+e)z^4 + fz^3 + s(3s+e)z^2 + s^3 = 0$

Now we can eliminate the terms in z^4 *and* z^2 *(since they have a common factor (3s+e)) to reduce the equation to a triquadratic equation (/disguised quadratic equation). If* $3s+e = 0$ *, then the coefficients of* z^4 *and* z^2 *will be equal to zero. This implies that* $s = -\frac{e}{3}$

When $s = -\frac{e}{3}$

$\Rightarrow z^6 + fz^3 - \frac{e^3}{27} = 0$

Substitution 3

This substitution is made to reduce a triquadratic equation to a quadratic equation

Let $w = z^3$

$\Rightarrow w^2 + fw - \frac{e^3}{27} = 0$

Now its easy to solve a quadratic equation

Using the method of completing the square

$\Rightarrow (w + \frac{f}{2})^2 - \frac{f^2}{4} - \frac{e^3}{27} = 0$

$\Rightarrow (w + \frac{f}{2})^2 = \frac{27f^2+4e^3}{108}$

$\Rightarrow w = -\frac{f}{2} \pm \sqrt{\frac{27f^2+4e^3}{108}}$

$\Rightarrow w_1 = -\frac{f}{2} + \sqrt{\frac{27f^2+4e^3}{108}}$ or $w_2 = -\frac{f}{2} - \sqrt{\frac{27f^2+4e^3}{108}}$

Notes

NB $\sim$ *We have* ***two*** *values of* w *ie* w_1 *and* w_2

If we use w_1 in making back substitutions
★ back sub for sub 3 ie $z = \sqrt[3]{w}$,(using De Moivres theorem) we get three values of z ie z_1 , z_2 and z_3
★ back sub for sub 2 ie $y = z + \frac{s}{z}$, substituting z_1 , z_2 and z_3 ,we get three values of y ie y_1 , y_2 and y_3
★ back sub for sub 1 ie $x = y - \frac{b}{3a}$, substituting y_1 , y_2 and y_3 ,we get three values of x ie x_1 , x_2 and x_3

If we use w_2
★ back sub for sub 3 ie $z = \sqrt[3]{w}$,(using De Moivres theorem) we get three values of z ie z_4 , z_5 and z_6
★ back sub for sub 2 ie $y = z + \frac{s}{z}$, substituting z_4 , z_5 and z_6 ,we get three values of y ie y_4 , y_5 and y_6
★ back sub for sub 1 ie $x = y - \frac{b}{3a}$, substituting y_4 , y_5 and y_6 ,we get three values of x ie x_4 , x_5 and x_6

Three values of x we get using w_1 (ie x_1 , x_2 and x_3) are the same values of x we get using w_2 (ie x_4 , x_5 and x_6). So we choose one value of w , either w_1 or w_2

Back substitutions using w_1

back substitution for sub 3

★ $w = z^3$

$\Rightarrow z = \sqrt[3]{w}$

$\Rightarrow z = \sqrt[3]{-\frac{f}{2} + \sqrt{\frac{27f^2+4e^3}{108}}}$

back substitution for sub 1

★$y = z + \frac{s}{z}$

$\Rightarrow y = \sqrt[3]{-\frac{f}{2} + \sqrt{\frac{27f^2+4e^3}{108}}} + \frac{-\frac{e}{3}}{\sqrt[3]{-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}}}$

But $\frac{-\frac{e}{3}}{\sqrt[3]{-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}}} = \sqrt[3]{-\frac{f}{2} - \sqrt{\frac{27f^2+4e^3}{108}}}$ *(Proof 1 , below)*

$$\Rightarrow y = \sqrt[3]{-\frac{f}{2} + \sqrt{\frac{27f^2+4e^3}{108}}} + \sqrt[3]{-\frac{f}{2} - \sqrt{\frac{27f^2+4e^3}{108}}}$$

back substitution for sub 1

$\bigstar x = y - \frac{b}{3a}$

$$\Rightarrow x = \sqrt[3]{-\frac{f}{2} + \sqrt{\frac{27f^2+4e^3}{108}}} + \sqrt[3]{-\frac{f}{2} - \sqrt{\frac{27f^2+4e^3}{108}}} - \frac{b}{3a}$$

Analogous formula

where $e = \frac{3ac-b^2}{3a^2}$ *and* $f = \frac{2b^3+27a^2d-9abc}{27a^3}$

Proof 1

Show that $\frac{-\frac{e}{3}}{\sqrt[3]{-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}}} = \sqrt[3]{-\frac{f}{2} - \sqrt{\frac{27f^2+4e^3}{108}}}$

OR | Show that $\frac{-\frac{e}{3}}{\sqrt[3]{w_1}} = \sqrt[3]{w_2}$

Solution

$$LHS = \frac{-\frac{e}{3}}{\sqrt[3]{-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}}}$$

multiply numerator and denominator by the conjugate of the denominator

$$LHS = \frac{-\frac{e}{3}\times\left(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}}\right)}{\sqrt[3]{-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}}\times\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}}}$$

$$LHS = \frac{-\frac{e}{3}\times(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}})}{\sqrt[3]{\left(-\frac{f}{2}+\sqrt{\frac{27f^2+4e^3}{108}}\right)\times\left(-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}\right)}}$$

$$LHS = \frac{-\frac{e}{3}\times(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}})}{\sqrt[3]{\frac{f^2}{4}-\frac{27f^2+4e^3}{108}}}$$

$$LHS = \frac{-\frac{e}{3}\times(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}})}{\sqrt[3]{\frac{27f^2-27f^2-4e^3}{108}}}$$

$$LHS = \frac{-\frac{e}{3}\times(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}})}{\sqrt[3]{\frac{-e^3}{27}}}$$

$$LHS = \frac{-\frac{e}{3}\times(\sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}})}{\frac{-e}{3}}$$

$$LHS = \sqrt[3]{-\frac{f}{2}-\sqrt{\frac{27f^2+4e^3}{108}}} \text{ (shown)}$$

4.1.1 Summary

Solve $ax^3 + bx^2 + cx + d = 0$

SOLUTION (SUMMARY)

1. Let $x = y - \frac{b}{3a}$ (sub 1)

$\sim y^3 + ey + f = 0$ (depressed cubic equation)

2. Let $y = z + \frac{s}{z}$ (sub 2) where s is a constant

$\sim z^6 + (3s+e)z^4 + fz^3 + s(3s+e)z^2 + s^3 = 0$

3. When $s = -\frac{e}{3}$

$\sim z^6 + fz^3 - \frac{e^3}{27} = 0$ (triquadratic equation)

4. Let $w = z^3$ (sub 3)

$\sim w^2 + fw - \frac{e^3}{27} = 0$ (quadratic equation)

5. Solve for w (solve the quadratic equation)

$\sim w = -\frac{f}{2} \pm \sqrt{\frac{27f^2+4e^3}{108}}$

6. Choose 1 value of w to be used in making back substitutions

$\sim w_1$

7. Back sub for sub 3

$\sim z = \sqrt[3]{w_1}$ (use De Moivres theorem to calculate 3 cube roots of w_1)

8. Back substitution for sub 2

$\sim y = z + \frac{s}{z} = \sqrt[3]{w_1} + \frac{-\frac{e}{3}}{\sqrt[3]{w_1}}$

9. $\frac{s}{z} = \frac{\frac{-e}{3}}{\sqrt[3]{w_1}} = \sqrt[3]{w_2}$ (proof 1)

$\sim y = \sqrt[3]{w_1} + \sqrt[3]{w_2}$

10. Back substitution for sub 1

$\sim x = y - \frac{b}{3a} = \underline{\underline{\sqrt[3]{w_1} + \sqrt[3]{w_2} - \frac{b}{3a}}}$

4.1.2 Example 1

Solve the equation $x^3 + x + 1 = 0$ using suitable substitutions

Solution

$x^3 + x + 1 = 0$

NB$\sim$ This is a depressed cubic equation (So ,there is no need to make the substitution $x = y - \frac{b}{3a}$)

Substitution 1

Let $x = z + \frac{s}{z}$ where s is a constant

$\Rightarrow (z + \frac{s}{z})^3 + (z + \frac{s}{z}) + 1 = 0$

$\Rightarrow (\frac{z^2+s}{z})^3 + (\frac{z^2+s}{z}) + 1 = 0$

$\Rightarrow \frac{z^6+3sz^4+3s^2z^2+s^3}{z^3} + \frac{z^2+s}{z} + 1 = 0$

multiply each term by z^3

$\Rightarrow (z^6 + 3sz^4 + 3s^2z^2 + s^3) + (z^4 + sz^2) + z^3 = 0$

$\Rightarrow z^6 + 3sz^4 + z^4 + z^3 + 3s^2z^2 + sz^2 + s^3 = 0$

$\Rightarrow z^6 + (3s + 1)z^4 + z^3 + s(3s + 1)z^2 + s^3 = 0$

When $s = -\frac{1}{3}$

$\Rightarrow z^6 + z^3 + (-\frac{1}{3})^3 = 0$

$\Rightarrow z^6 + z^3 - \frac{1}{27} = 0$

Substitution 2

Let $w = z^3$

$\Rightarrow w^2 + w - \frac{1}{27} = 0$

Solve the quadratic equation

$\Rightarrow (w + \frac{1}{2})^2 - \frac{1}{4} - \frac{1}{27} = 0$

$\Rightarrow (w + \frac{1}{2})^2 = \frac{31}{108}$

$\Rightarrow w = -\frac{1}{2} \pm \sqrt{\frac{31}{108}}$

$\Rightarrow w_1 = -\frac{1}{2} + \sqrt{\frac{31}{108}}$ and $w_2 = -\frac{1}{2} - \sqrt{\frac{31}{108}}$

Back substitutions (Using w_1)

back substitution for sub 2

★$w = z^3$

$\Rightarrow z = \sqrt[3]{w_1}$

Let $w_1 = r(cos\theta + isin\theta)$

where $r = |w_1| = -\frac{1}{2} + \sqrt{\frac{31}{108}}$ *and* $\theta = arg(w_1) = 0$

$\Rightarrow z_{k+1} = \sqrt[3]{w_1} = r^{\frac{1}{3}}\left(cos(\frac{\theta+2k\pi}{3}) + isin(\frac{\theta+2k\pi}{3})\right)$ for $k = 0, 1, 2$

$\Rightarrow z_{k+1} = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}\left(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\right)$ for $k = 0, 1, 2$

Substitute $k = 0, 1, 2$

when $k = 0$

$\Rightarrow z_1 = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}\left(cos(0) + isin(0)\right) = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{31}{108}}}$

When $k = 1$

$\Rightarrow z_2 = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}\left(cos(\frac{2\pi}{3}) + isin(\frac{2\pi}{3})\right) = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

When $k = 2$

$$\Rightarrow z_3 = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}\Big(cos(\frac{4\pi}{3}) + isin(\frac{4\pi}{3})\Big) = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$$

back substitution for sub 1

★$x = z + \frac{s}{z}$

$$\Rightarrow x = z + \frac{-\frac{1}{3}}{z}$$

substitute z_1 , z_2 *and* z_3

$$\Rightarrow x_1 = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{31}{108}}} + \frac{-\frac{1}{3}}{\sqrt[3]{-\frac{1}{2}+\sqrt{\frac{31}{108}}}}$$

$$\Rightarrow x_2 = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) + \frac{-\frac{1}{3}}{\sqrt[3]{(-\frac{1}{2}+\sqrt{\frac{31}{108}})}(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}$$

$$\Rightarrow x_3 = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) + \frac{-\frac{1}{3}}{\sqrt[3]{(-\frac{1}{2}+\sqrt{\frac{31}{108}})}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)}$$

$$\Rightarrow \underline{\underline{x_{\mathbf{1}} = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{31}{108}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{31}{108}}}}}$$

$$\underline{\underline{x_{\mathbf{2}} = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) + \sqrt[3]{(-\frac{1}{2} - \sqrt{\frac{31}{108}})}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)}} \textbf{ see } note^{\mathbf{2}}$$

$$\underline{\underline{x_{\mathbf{3}} = \sqrt[3]{(-\frac{1}{2} + \sqrt{\frac{31}{108}})}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) + \sqrt[3]{(-\frac{1}{2} - \sqrt{\frac{31}{108}})}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)}} \textbf{ see } note^{\mathbf{3}}$$

This is the exact solution of the equation $x^3 + x + 1 = 0$

Note

$\sim x_1 \approx -0.6823278038280193$
$\sim x_2 \approx 0.34116390191 + 1.1615414i$
$\sim x_3 \approx 0.34116390191 - 1.1615414i$

1.A cubic equation has at least 1 real root(x_1 is a real root in example 1)
2.If a cubic equation has complex roots , then the complex roots are conjugates (x_2 and x_3 are conjugates)

$note^2 \sim \sqrt[3]{(-\frac{1}{2}+\sqrt{\frac{31}{108}})}(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ and $\sqrt[3]{(-\frac{1}{2}-\sqrt{\frac{31}{108}})}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ are not complex conjugates.

$note^3 \sim \sqrt[3]{(-\frac{1}{2}+\sqrt{\frac{31}{108}})}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ and $\sqrt[3]{(-\frac{1}{2}-\sqrt{\frac{31}{108}})}(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ are not complex conjugates

4.1.3 Example 2

Solve the equation $x^3 + x^2 + x + 1 = 0$

Solution

Substitution 1

textitLet $x = y - \frac{b}{3a} = y - \frac{1}{3}$

$\Rightarrow (y-\frac{1}{3})^3 + (y-\frac{1}{3})^2 + (y-\frac{1}{3}) + 1 = 0$

$\Rightarrow (\frac{3y-1}{3})^3 + (\frac{3y-1}{3})^2 + (\frac{3y-1}{3}) + 1 = 0$

$\Rightarrow (\frac{27y^3-27y^2+9y-1}{27}) + (\frac{9y^2-6y+1}{9}) + (\frac{3y-1}{3}) + 1 = 0$

multiply each term by 27

$\Rightarrow (27y^3 - 27y^2 + 9y - 1) + (27y^2 - 18y + 3) + 27y - 9 + 27 = 0$

$\Rightarrow 27y^3 + 18y + 20 = 0$

Substitution 2

Let $y = z + \frac{s}{z}$ *where s is a constant*

$\Rightarrow 27(z+\frac{s}{z})^3 + 18(z+\frac{s}{z}) + 20 = 0$

$\Rightarrow 27(\frac{z^2+s}{z})^3 + 18(\frac{z^2+s}{z}) + 20 = 0$

$\Rightarrow 27(\frac{z^6+3sz^4+3s^2z^2+s^3}{z^3}) + \frac{18z^2+18s}{z} + 20 = 0$

$\Rightarrow 27z^6 + 81sz^4 + 81s^2z^2 + 27s^3 + 18z^4 + 18sz^2 + 20z^3 = 0$

$\Rightarrow 27z^6 + (81s+18)z^4 + 20z^3 + s(81s+18) + 27s^3 = 0$

When $s = -\frac{2}{9}$

$\Rightarrow 27z^6 + 20z^3 + 27(-\frac{2}{9})^3 = 0$

$\Rightarrow 27z^6 + 20z^3 - \frac{8}{27} = 0$

Substitution 3

Let $w = y^3$

$\Rightarrow 27w^2 + 20w - \frac{8}{27} = 0$

Solve the quadratic equation

$\Rightarrow w = -\frac{10}{27} \pm \frac{\sqrt{108}}{27}$

$\Rightarrow w_1 = -\frac{10}{27} + \frac{\sqrt{108}}{27}$ and $w_2 = -\frac{10}{27} - \frac{\sqrt{108}}{27}$

Back substitutions (Using w_1)

back substitution for sub 3

★$w = z^3$

$z = \sqrt[3]{w_1}$

Let $w_1 = r(cos\theta + isin\theta)$

$\sim w_1$ *is a positive real number and thus,*

$r = |w_1| = -\frac{10}{27} + \frac{\sqrt{108}}{27}$ *and* $\theta = arg(w_1) = 0$

$\Rightarrow z_{k+1} = \sqrt[3]{w_1} = r^{\frac{1}{n}}\left(cos(\frac{\theta+2k\pi}{3}) + isin(\frac{\theta+2k\pi}{3})\right)$ for $k = 0, 1, 2$

$$\Rightarrow z_{\text{k+1}} = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(cos(\tfrac{2k\pi}{3}) + isin(\tfrac{2k\pi}{3})\right) \text{ for } k = 0, 1, 2$$

Substitute $k = 0, 1, 2$

$$\Rightarrow z_1 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}(cos0 + isin0)$$

$$\Rightarrow z_2 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(cos(\tfrac{2\pi}{3}) + isin(\tfrac{2\pi}{3})\right)$$

$$\Rightarrow z_3 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(cos(\tfrac{4\pi}{3}) + isin(\tfrac{4\pi}{3})\right)$$

$$\Rightarrow z_1 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}$$

$$\Rightarrow z_2 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)$$

$$\Rightarrow z_3 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)$$

back substitution for sub 2

$$\bigstar y = z + \frac{s}{z} = z + \frac{-\frac{2}{9}}{z}$$

$$\Rightarrow y_1 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}} + \frac{-\frac{2}{9}}{\sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}}$$

$$\Rightarrow y_2 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) + \frac{-\frac{2}{9}}{\sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}$$

$$\Rightarrow y_3 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) + \frac{-\frac{2}{9}}{\sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)}$$

$$\Rightarrow y_1 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}} + \sqrt[3]{-\frac{10}{27} - \frac{\sqrt{108}}{27}}$$

$$\Rightarrow y_2 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) + \sqrt[3]{-\frac{10}{27} - \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)$$

$$\Rightarrow y_3 = \sqrt[3]{-\frac{10}{27} + \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) + \sqrt[3]{-\frac{10}{27} - \frac{\sqrt{108}}{27}}\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)$$

$\Rightarrow y_1 = -\frac{2}{3}$

$\Rightarrow y_2 = \frac{1}{3} + i$

$\Rightarrow y_3 = \frac{1}{3} - i$

back substitution for sub 1

$\bigstar x = y - \frac{1}{3}$

$\Rightarrow x_1 = -\frac{2}{3} - \frac{1}{3}$

$\Rightarrow x_2 = \frac{1}{3} + i - \frac{1}{3}$

$\Rightarrow x_3 = \frac{1}{3} - i - \frac{1}{3}$

$\Rightarrow \underline{\underline{x_1 = -1 \text{ , } x_2 = i \text{ , } x_3 = -i}}$

4.1.4 Example 3

Solve the equation $x^3 + x^2 - 10x + 8 = 0$ using suitable substitutions

Solution

$x^3 + x^2 - 10x + 8 = 0$

Substitution 1

Let $x = y - \frac{b}{3a} = y - \frac{1}{3}$

$\Rightarrow (y - \frac{1}{3})^3 + (y - \frac{1}{3})^2 - 10(y - \frac{1}{3}) + 8 = 0$

$\Rightarrow (\frac{3y-1}{3})^3 + (\frac{3y-1}{3})^2 - 10(\frac{3y-1}{3}) + 8 = 0$

$\Rightarrow (\frac{27y^3-27y^2+9y-1}{27}) + (\frac{9y^2-6y+1}{9}) - (\frac{30y-10}{3}) + 8 = 0$

$\Rightarrow (27y^3 - 27y^2 + 9y - 1) + (27y^2 - 18y + 3) - (270y - 90) + 216 = 0$

$\Rightarrow 27y^3 - 279y + 308 = 0$

Substitution 2

Let $y = z + \frac{s}{z}$

$\Rightarrow 27(z + \frac{s}{z})^3 - 279(z + \frac{s}{z}) + 308 = 0$

$\Rightarrow 27(\frac{z^2+s}{z})^3 - 279(\frac{z^2+s}{z}) + 308 = 0$

$\Rightarrow 27(\frac{z^6+3sz^4+3s^2z^2+s^3}{z^3}) - 279(\frac{z^2+s}{z}) + 308 = 0$

$\Rightarrow 27(z^6 + 3sz^4 + 3s^2z^2 + s^3) - 279(z^4 + sz^2) + 308z^3 = 0$

$\Rightarrow 27z^6 + (81s - 279)z^4 + 308z^3 + s(81s - 279)z^2 + 27s^3 = 0$

When $s = \frac{31}{9}$

$\Rightarrow 27z^6 + 308z^3 + 27(\frac{31}{9})^3 = 0$

$\Rightarrow 27z^6 + 308z^3 + \frac{29791}{27} = 0$

Substitution 3

Let $w = z^3$

$\Rightarrow 27w^2 + 308w + \frac{29791}{27} = 0$

Solve the quadratic equation

$\Rightarrow w = -\frac{154}{27} \pm \sqrt{-\frac{25}{3}}$

$\Rightarrow w = -\frac{154}{27} \pm \frac{5\sqrt{3}i}{3}$

$\Rightarrow w_1 = -\frac{154}{27} + \frac{5\sqrt{3}i}{3}$ and $w_2 = -\frac{154}{27} - \frac{5\sqrt{3}i}{3}$

Back substitutions (Using w_1)

back substitution for sub 3

$z = \sqrt[3]{w_1}$

Let $w_1 = r(cos\theta + isin\theta)$

$\sim r = |w_1| = \frac{31\sqrt{31}}{27}$ and $\theta = arg(w_1) = \pi - \arctan\frac{45\sqrt{3}}{154}$

$\Rightarrow z_{\text{k+1}} = \sqrt[3]{w_1} = r^{\frac{1}{3}}\left(cos(\frac{\theta+2k\pi}{3}) + isin(\frac{\theta+2k\pi}{3})\right)$
for $k = 0, 1, 2$

$\Rightarrow z_{\text{k+1}} = \sqrt[3]{\frac{31\sqrt{31}}{27}}\left(cos(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3}) + isin(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})\right)$
for $k = 0, 1, 2$

Substitute $k = 0, 1, 2$
$\gg z_1 = \frac{7}{6} + 1.44337567297i$

$z_2 = -\frac{11}{6} + 0.28867513459i$

$z_3 = \frac{2}{3} - 1.73205080757i$

NB $\sim$ The complex/imaginary parts of all the three values of z are not exact. If we use these values, we will not get the exact roots , so we must substitute $z_{\text{k+1}}$ directly without substituting k and we will substitute k later.

back substitution for sub 2

★$y = z + \frac{\frac{31}{9}}{z}$

$$\Rightarrow y_{\text{k+1}} = \sqrt[3]{\frac{31\sqrt{31}}{27}}\left(cos(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3}) + isin(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})\right)..$$
$$.. + \frac{\frac{31}{9}}{\sqrt[3]{\frac{31\sqrt{31}}{27}}\left(cos(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})+isin(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})\right)}$$

$$\Rightarrow y_{\text{k+1}} = \sqrt[3]{\frac{31\sqrt{31}}{27}}\left(cos(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3}) + isin(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})\right)..$$
$$.. + \sqrt[3]{\frac{31\sqrt{31}}{27}}\left(cos(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3}) - isin(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3})\right)$$
for $k = 0, 1, 2$

$$\Rightarrow y_{\text{k+1}} = 2\left(\sqrt[3]{\frac{31\sqrt{31}}{27}}\right)cos\left(\frac{\pi-\arctan\frac{45\sqrt{3}}{154}+2k\pi}{3}\right)$$

back substitution for sub 1

★$x = y - \frac{1}{3}$

$\Rightarrow x_{\text{k+1}} = 2\left(\sqrt[3]{\frac{31\sqrt{31}}{27}}\right)cos(\frac{\pi - \arctan\frac{45\sqrt{3}}{154} + 2k\pi}{3}) - \frac{1}{3}$
for $k = 0, 1, 2$

Substitute $k = 0, 1, 2$

$\Rightarrow x_1 = 2\left(\sqrt[3]{\frac{31\sqrt{31}}{27}}\right)cos(\frac{\pi - \arctan\frac{45\sqrt{3}}{154}}{3}) - \frac{1}{3}$

$x_2 = 2\left(\sqrt[3]{\frac{31\sqrt{31}}{27}}\right)cos(\frac{\pi - \arctan\frac{45\sqrt{3}}{154} + 2\pi}{3}) - \frac{1}{3}$

$x_3 = 2\left(\sqrt[3]{\frac{31\sqrt{31}}{27}}\right)cos(\frac{\pi - \arctan\frac{45\sqrt{3}}{154} + 4\pi}{3}) - \frac{1}{3}$

$\Rightarrow x_1 = 2\frac{1}{3} - \frac{1}{3}$

$x_2 = -3\frac{2}{3} - \frac{1}{3}$

$x_3 = 1\frac{1}{3} - \frac{1}{3}$

$\Rightarrow$ $x_1 = 2$, $x_2 = -4$ and $x_3 = 1$

4.1.5 Example 4

Solve $2x^3 - 12x^2 - 9x - 35 = 0$

Solution

$2x^3 - 12x^2 - 9x - 35 = 0$

Substitution 1

Let $x = y - \frac{b}{3a} = y - \frac{-12}{3(2)} = y + 2$

$\Rightarrow 2(y+2)^3 - 12(y+2)^2 - 9(y+2) - 35 = 0$

$\Rightarrow 2(y^3 + 6y^2 + 12y + 8) - 12(y^2 + 4y + 4) - 9(y+2) - 35 = 0$

$\Rightarrow 2y^3 - 33y - 85 = 0$

Substitution 2

Let $y = z + \frac{s}{z}(sub2)$

$\Rightarrow 2(z + \frac{s}{z})^3 - 33(z + \frac{s}{z}) - 85 = 0$

$\Rightarrow 2(\frac{z^2+s}{z})^3 - 33(\frac{z^2+s}{z}) - 85 = 0$

$\Rightarrow 2(\frac{z^6+3sz^4+3s^2z^2+s^3}{z^3}) - 33(\frac{z^2+s}{z}) - 85 = 0$

multiply each term by z^3

$\Rightarrow (2z^6 + 6sz^4 + 6s^2z^2 + 2s^3) - (33z^4 + 33sz^2) - 85z^3 = 0$

$\Rightarrow 2z^6 + (6s - 33)z^4 - 85z^3 + s(6s - 33)z^2 + 2s^3 = 0$

When $s = \frac{11}{2}$

$\Rightarrow 2z^6 - 85z^3 + 2(\frac{11}{2})^3 = 0$

$\Rightarrow 2z^6 - 85z^3 + \frac{1331}{4} = 0$

Substitution 3

Let $w = z^3$

$\Rightarrow 2w^2 - 85w + \frac{1331}{4} = 0$

Solve the quadratic equation

$\Rightarrow w = \frac{85}{4} \pm \frac{39\sqrt{3}}{4}$

$\Rightarrow w_1 = \frac{85}{4} + \frac{39\sqrt{3}}{4}$ and $w_2 = \frac{85}{4} - \frac{39\sqrt{3}}{4}$

Back substitutions (Using w_1)

back substitution for sub 3

★$w = z^3$

$z = \sqrt[3]{w_1} = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}$

$\sim r = |w_1| = \frac{85}{4} + \frac{39\sqrt{3}}{4}$ and $\theta = arg(w_1) = 0$

$z_{k+1} = r^{\frac{1}{3}}\left(cos(\frac{\theta+2k\pi}{3}) + isin(\frac{\theta+2k\pi}{3})\right)$

$$z_{k+1} = \sqrt[3]{(\frac{85}{4} + \frac{39\sqrt{3}}{4})}\Big(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\Big)$$

back substitution for sub 1

$\bigstar y = z + \frac{\frac{11}{2}}{z}$

$$\Rightarrow y_{k+1} = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\Big) + \frac{\frac{11}{2}}{\sqrt[3]{(\frac{85}{4} + \frac{39\sqrt{3}}{4})}\Big(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\Big)}$$

$$\Rightarrow y_{k+1} = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2k\pi}{3}) - isin(\frac{2k\pi}{3})\Big)$$

back substitution for sub 1

$\bigstar x = y + 2$

$$\Rightarrow x_{k+1} = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2k\pi}{3}) + isin(\frac{2k\pi}{3})\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2k\pi}{3}) - isin(\frac{2k\pi}{3})\Big) + 2$$

for $k = 0, 1, 2$

Substitute $k = 0, 1, 2$

$$\Rightarrow x_1 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos0 + isin0\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(cos0 - isin0\Big) + 2$$

$$\Rightarrow x_2 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2\pi}{3}) + isin(\frac{2\pi}{3})\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(cos(\frac{2\pi}{3}) - isin(\frac{2\pi}{3})\Big) + 2$$

$$\Rightarrow x_3 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(cos(\frac{4\pi}{3}) + isin(\frac{4\pi}{3})\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(cos(\frac{4\pi}{3}) - isin(\frac{4\pi}{3})\Big) + 2$$

$$\Rightarrow x_1 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}} + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}} + 2$$

$$\Rightarrow x_2 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) + 2$$

$$\Rightarrow x_3 = \sqrt[3]{\frac{85}{4} + \frac{39\sqrt{3}}{4}}\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) + \sqrt[3]{\frac{85}{4} - \frac{39\sqrt{3}}{4}}\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) + 2$$

$$\Rightarrow x_1 = 5 + 2$$

$$\Rightarrow x_2 = -\frac{5}{2} + \frac{3}{2}i + 2$$

$\Rightarrow x_3 = -\frac{5}{2} - \frac{3}{2}i + 2$

$$\Rightarrow \underline{\underline{x_1 = 7 \quad , \quad x_2 = -\frac{1}{2} + \frac{3}{2}i \quad , \quad x_3 = -\frac{1}{2} - \frac{3}{2}i}}$$

4.1.6 EXERCISE 2

1. Solve the following equations using suitable substitutions
   (a) $x^3 + x^2 + 1 = 0$
   (b) $x^3 - 8x^2 - 3x + 90 = 0$
   (c) $2x^3 - 7x^2 + 3x + 5 = 0$
   (d) $3x^3 - 2x^2 - 2x - 5 = 0$
   (e) $2x^3 - 21x - 9 = 0$
   (f) $x^3 + 1 = 0$
2. Find the general solutions to the following equations
   (a) $x^3 + ax + b$
   (b) $x^3 + ax^2 + b = 0$
   (c) $x^3 + bx^2 - a^2x + a^2b = 0$

5 SUMMARY

The table below show the substitutions used in solving linear, quadratic and cubic equations, maximum number of roots and solutions.

	Equation	Sub 1	Sub 2	Sub 3	Number of roots	Solution
1	Linear	$x = y - \frac{b}{a}$	—	—	1	$x = -\frac{b}{a}$
2	Quadratic	$x = y - \frac{b}{2a}$	$w = y^2$	—	2	$x = \sqrt{w} - \frac{b}{2a}$
3	Cubic	$x = y - \frac{b}{3a}$	$y = z + \frac{s}{z}$	$w = z^3$	3	$x = \sqrt[3]{w_1} + \sqrt[3]{w_2} - \frac{b}{3a}$

You can visit our website http://womcalculator.droppages.com to solve any linear, quadratic and cubic equations using WOMCALCULATOR.

WORLD OF MATHEMATICS

Mathematics Made Easier

http://worldofmathematics.droppages.com

womaths@gmail.com