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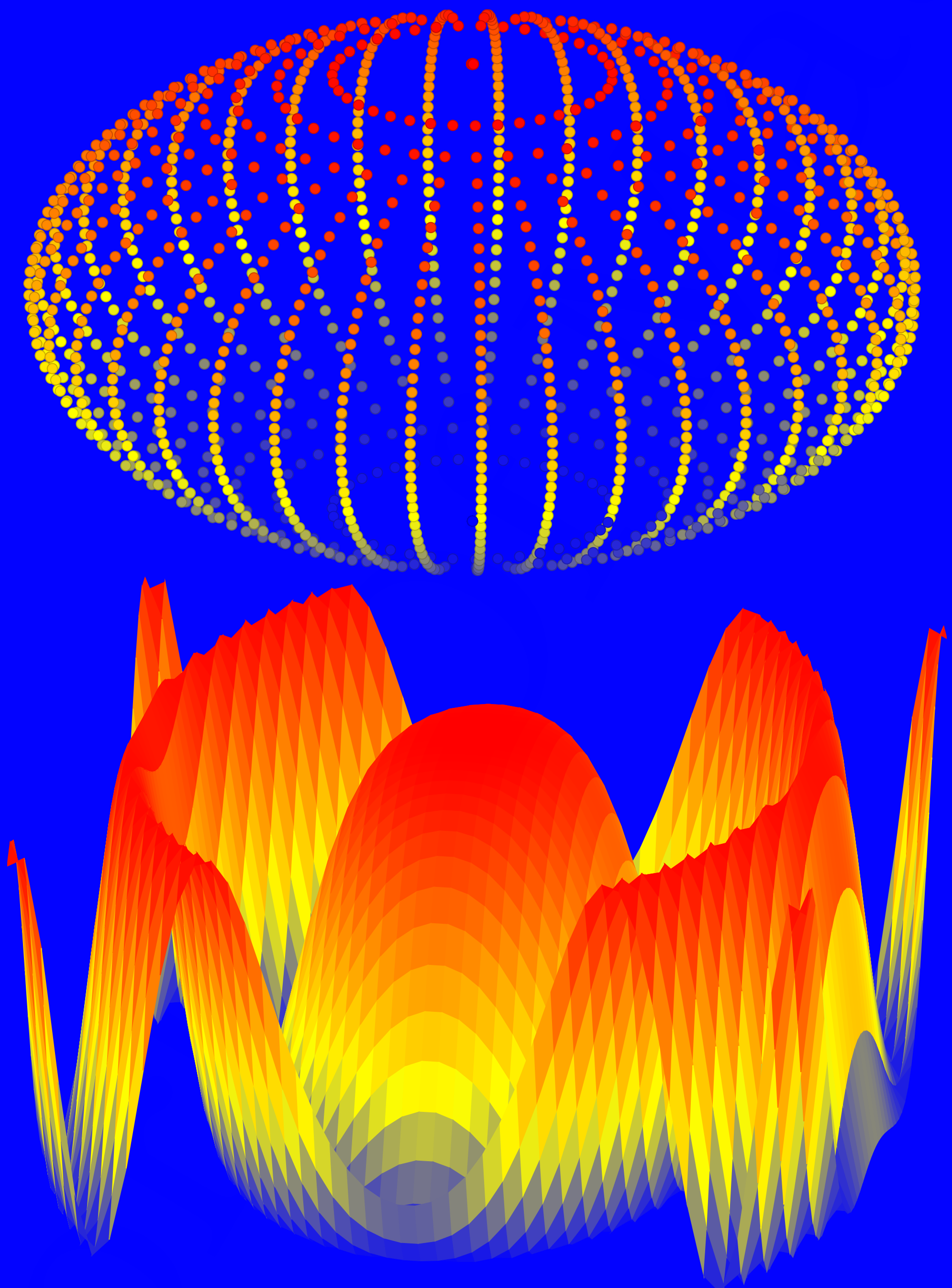

★ PROOF BY MATHEMATICAL INDUCTION ★

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WORLD OF MATHEMATICS TUTORIALS

PROOF BY MATHEMATICAL INDUCTION

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November 24, 2020

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Chapter 1

MATHEMATICAL INDUCTION

1.1 Introduction

Mathematical induction is a technique for proving a statement that is asserted about every natural number. A statement can be a theorem or a formula.

There are 4 steps that are required in a mathematical induction proof which are :

- **Initial Step**
  Show that the statement is true for the first natural number satisfied by the statement which is $n = 1$ if the statement is true for all $n \in \mathbb{N}$ .Alternatively , we can show that the statement is true for the first 2 or 3 natural numbers satisfied by the statement.

- **Assumption Step**
  Assume that the statement holds for $n = k$ where $k \in \mathbb{N}$. This is also called the inductive hypothesis .

- **Inductive step**
  This is the most important step in a proof by mathematical induction. We have to show that the statement is true for $n = k + 1$ and we must apply the results of the assumption step in the inductive step [i.e we should not show by direct substitution (substituting $k + 1$ for n) , but direct substitution is used to verify our answer and should not be shown in our working]

- **Conclusion**
  Conclude that the statement is true for all $n \in \mathbb{N}$ since its true for $n = 1$, $\ldots$, for $n = k$ and for $n = k + 1$. [If a statement is true for $n = 1$, $n = k$ & $n = k + 1$ for all $k \in \mathbb{N}$, then it implies that a statement is true for $n = 1, 2, 3, 4, \ldots$ ie for all $n \in \mathbb{N}$]

Proof by induction is used to prove different types of mathematical statements . In this book , we are going to group them as follows :

- Summations
- Products
- Derivatives

- Divisions
- inequalities

NB $\sim$ *these are not all mathematical statements.*

Please note that if the range of n is not given in a question , then $n \in \mathbb{N}$

1.2 Summations

Firstly , we need to understand the sigma notation (summation notation). Sigma notation has 4 parts i.e the sign (Σ) , lower limit (below the sign) , upper limit (above the sign) and the function (just after sign) as shown below

$$\sum_{r=L.lim}^{U.lim} f(r)$$

For example $\sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$.

When n is small , it is easy to compute the sums by adding the terms of r^2 but when n is large , it is difficult to compute the sum .For example , if $n = 10\ 000$ you can take 4-5 hours adding the terms of r^2 from $r = 1$ to $r = 10000$. So if we can find a general formula for the sum of the first n terms , then we can compute the sum in seconds.

The general formula of $\sum_{r=1}^{n} r^2$ can be derived using method of differences as shown below .

Deriving the formula for $\sum_{r=1}^{n} r^2$ using the method of differences

$\Rightarrow r(r+1) = a[r(r+1)(r+2) - (r-1)r(r+1)]$

$$\Rightarrow r(r+1) = ar(r+1)[(r+2) - (r-1)]$$

$$\Rightarrow r(r+1) = 3ar(r+1)$$

$$\Rightarrow \underline{a = \tfrac{1}{3}}$$

$\therefore r(r+1) = \frac{1}{3}[r(r+1)(r+2) - (r-1)r(r+1)]$

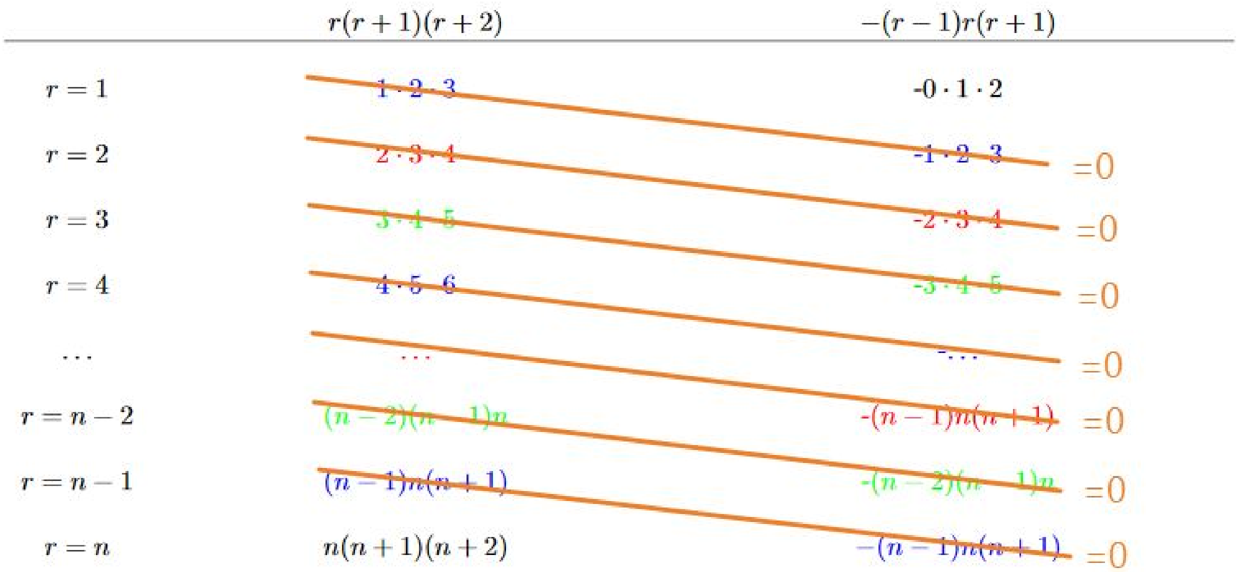
Introducing the sums on both sides

$\Rightarrow \sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} \frac{1}{3}[r(r+1)(r+2) - (r-1)r(r+1)]$

$\Rightarrow \sum_{r=1}^{n}(r^2+r) = \frac{1}{3}\sum_{r=1}^{n}[r(r+1)(r+2) - (r-1)r(r+1)]$

★ Find $\sum_{r=1}^{n} r(r+1)(r+2) - (r-1)r(r+1)$ using the method of differences as shown **below**

	$r(r+1)(r+2)$	$-(r-1)r(r+1)$
$r=1$	$1\cdot2\cdot3$	$-0\cdot1\cdot2$
$r=2$	$2\cdot3\cdot4$	$-1\cdot2\cdot3$
$r=3$	$3\cdot4\cdot5$	$-2\cdot3\cdot4$
$r=4$	$4\cdot5\cdot6$	$-3\cdot4\cdot5$
$\dots$	$\dots$	$-\dots$
$r=n-2$	$(n-2)(n-1)n$	$-(n-1)n(n+1)$
$r=n-1$	$(n-1)n(n+1)$	$-(n-2)(n-1)n$
$r=n$	$n(n+1)(n+2)$	$-(n-1)n(n+1)$



★ $\therefore \sum_{r=1}^{n}[r(r+1)(r+2) - (r-1)r(r+1)] = -0\cdot1\cdot2 + n(n+1)(n+2) =$ **n(n+1)(n+2)**

$\Rightarrow \sum_{r=1}^{n}(r^2+r) = \frac{1}{3}n(n+1)(n+2)$

$\Rightarrow \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{1}{3}n(n+1)(n+2)$

$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{1}{3}n(n+1)(n+2) - \sum_{r=1}^{n} r$

$\sum_{r=1}^{n} r$ is the sum of the first n terms in an arithmetic progression with $a=1$ and $d=1$, and therefore

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

$\Rightarrow \sum_{r=1}^{n} r^2 = \frac{1}{3}n(n+1)(n+2) - \frac{n}{2}(n+1) = \frac{1}{6}n(n+1)[2(n+2)-3]$

$\Rightarrow \underline{\underline{\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)}}$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

♦ **Using the general formula above ,**

- $\sum_{r=1}^{5} r^2 = \frac{5}{6}(5+1)(2\cdot 5+1) = \underline{\underline{55}}$,and
- $\sum_{r=1}^{10\ 000} r^2 = \frac{10\ 000}{6}(10\ 000+1)(2\cdot 10\ 000+1) = \underline{\underline{333\ 383\ 335\ 000}}$

We derive the general formula for $\sum_{r=1}^{n} r^2$ but we are not sure if its true for all $n \in \mathbb{N}$. May be its true for the first 100 terms only.

So we use proof by mathematical induction to prove that the general formula is true for all $n \in \mathbb{N}$ as shown below.

Example 1

Prove by mathematical induction that

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

Solution

Initial Step

When n=1

$LHS = \sum_{r=1}^{1} r^2 = 1^2 = \mathbf{1}$

$RHS = \frac{1}{6}(1+1)(2\times 1+1) = \mathbf{1}$

$\therefore LHS = RHS \sim$ *the statement is true for* $n = 1$

When n=2

$LHS = \sum_{r=1}^{2} r^2 = 1^2 + 2^2 = \mathbf{5}$

$RHS = \frac{2}{6}(2+1)(2 \times 2 + 1) = \mathbf{5}$

$\therefore LHS = RHS \sim$ *the statement is true for* $n = 2$

When $n = 3$

$LHS = \sum_{r=1}^{3} r^2 = 1^2 + 2^2 + 3^2 = \mathbf{14}$

$RHS = \frac{3}{6}(3+1)(2 \times 3 + 1) = \mathbf{14}$

$\therefore LHS = RHS$ *the statement is true for* $n = 3$

Assumption Step

Assume the statement holds for $n = k$ for all $k \in \mathbb{N}$.

$$\Rightarrow \qquad \sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$$

NB $\sim$ *this is direct substitution and we should not use it in the inductive step*

Inductive step

When $n = k + 1$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = [1^2 + 2^2 + 3^2 + \ldots + k^2] + (k+1)^2$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = \left[\sum_{r=1}^{k} r^2\right] + (k+1)^2$

But $\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$ *from assumption step*

$\Rightarrow \sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = (k+1)[\frac{k(2k+1)}{6} + (k+1)]$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = (k+1)[\frac{k(2k+1)+6(k+1)}{6}]$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = (k+1)[\frac{2k^2+7k+6}{6}]$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}[2k^2+3k+4k+6]$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}(k+2)(2k+3)$

$\Rightarrow \sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}(k+1+1)[2(k+1)+1]$

Thus , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k+1$

Notes

★ Given that $\sum_{r=1}^{n} f(r) = g(n)$,then $\sum_{r=1}^{k+1} f(r)$ (in the inductive step) is found using the following identity

$$\sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$$

and the result should be the same as the result we get by direct substitution i.e g(k+1) but direct substitution should not be used in the inductive step.

Proof

$$\begin{aligned} \Rightarrow \sum_{r=1}^{k+1} f(r) &= f(1) + f(2) + f(3) + \ldots + f(k-1) + f(k) + f(k+1) \\ &= [f(1) + f(2) + f(3) + \ldots + f(k-1) + f(k)] + f(k+1) \\ &= \sum_{r=1}^{k} f(r) + f(k+1) \end{aligned}$$

Example 2

Prove by induction that

$$\sum_{r=1}^{n} r^3 - 1 = \frac{n}{4}(n-1)(n^2+3n+4)$$

Solution

Initial Step

When $n = 1$

$LHS = 1^3 - 1 = \mathbf{0}$

$RHS = \frac{1}{4}(1-1)(1^2 + 3 \times 1 + 4) = \mathbf{0}$

$\therefore LHS = RHS$ *the statement is true for n=1*

When $n = 2$

.

$LHS = [1^3 - 1] + [2^3 - 1] = 0 + 7 = \mathbf{7}$

$RHS = \frac{2}{4}(2-1)(2^2 + 3 \times 2 + 4) = \frac{2 \times 1 \times 14}{4} = \mathbf{7}$

$\therefore LHS = RHS$ *the statement is true for n=2*

When $n = 3$

$LHS = [1^3 - 1] + [2^3 - 1] + [3^3 - 1] = 0 + 7 + 26 = \mathbf{33}$

$RHS = \frac{3}{4}(3-1)(3^2 + 3 \times 3 + 4) = \frac{3 \times 4 \times 22}{4} = \mathbf{33}$

$\therefore LHS = RHS$ *the statement is true for n=3*

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\sum_{r=1}^{k} r^3 - 1 = \frac{k}{4}(k-1)(k^2 + 3k + 4)$$

Inductive step

When $n = k + 1$

$\Rightarrow \sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$

NB $\sim f(r) = r^3 - 1$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \left[\sum_{r=1}^{k} r^3 - 1\right] + [(k+1)^3 - 1]$

But $\sum_{r=1}^{k} r^3 - 1 = \frac{k}{4}(k-1)(k^2 + 3k + 4)$ *from Assumption step*

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \frac{k}{4}(k-1)(k^2+3k+4) + [(k+1)^3 - 1]$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \frac{k(k-1)(k^2+3k+4)}{4} + \frac{k^3+3k^2+3k+1-1}{1}$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \frac{k(k-1)(k^2+3k+4)}{4} + \frac{k^3+3k^2+3k}{1}$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = k\left[\frac{(k-1)(k^2+3k+4)}{4} + \frac{k^2+3k+3}{1}\right]$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = k\left[\frac{(k-1)(k^2+3k+4)+4(k^2+3k+3)}{4}\right]$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = k\left[\frac{(k^3+3k^2+4k-k^2-3k-4)+(4k^2+12k+12)}{4}\right]$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = k\left[\frac{k^3+6k^2+13k+8}{4}\right]$

NB ~ *We want to factorise* $h(k) = k^3 + 6k^2 + 13k + 8$ *. Using factor theorem ,*

$$h(-1) = 1^3 + 6(-1)^2 + 13(1) + 8 = 0$$

and therefore $(k+1)$ *is a factor of* $h(k)$*. Devide* $h(k)$ *by* $(k+1)$ *using long division to get another factor of* $h(k)$ *which is* $k^2 + 5k + 8$*.*

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = k\left[\frac{(k+1)(k^2+5k+8)}{4}\right]$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \frac{(k+1)}{4}(k)(k^2+5k+8)$

$\Rightarrow \sum_{r=1}^{k+1} r^3 - 1 = \frac{(k+1)}{4}(k+1-1)[(k+1)^2 + 3(k+1) + 4]$

NB ~ *The last stage is the same as* $g(k+1)$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k+1$

Example 3

Prove by using the principle of mathematical induction that

$$\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1} = \frac{n}{n+1}$$

Solution

Initial Step

When $n = 1$

$LHS = \frac{1}{1} - \frac{1}{1+1} = \frac{1}{2}$

$RHS = \frac{1}{1+1} = \frac{1}{2}$

$\therefore LHS = RHS$ *the statement is true for n=1*

When $n = 2$

.

$LHS = \left[\frac{1}{1} - \frac{1}{1+1}\right] + \left[\frac{1}{2} - \frac{1}{2+1}\right] = \frac{2}{3}$

$RHS = \frac{2}{2+1} = \frac{2}{3}$

$\therefore LHS = RHS$ *the statement is true for n=2*

When $n = 3$

$LHS = \left[\frac{1}{1} - \frac{1}{1+1}\right] + \left[\frac{1}{2} - \frac{1}{2+1}\right] + \left[\frac{1}{3} - \frac{1}{3+1}\right] = \frac{3}{4}$

$RHS = \frac{3}{3+1} = \frac{3}{4}$

$\therefore LHS = RHS$ *the statement is true for n=3*

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\sum_{r=1}^{k} \frac{1}{r} - \frac{1}{r+1} = \frac{k}{k+1}$$

Inductive step

When $n = k + 1$

$\Rightarrow \sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \left[\sum_{r=1}^{k} \frac{1}{r} - \frac{1}{r+1}\right] + \frac{1}{k+1} - \frac{1}{k+1+1}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \left[\sum_{r=1}^{k} \frac{1}{r} - \frac{1}{r+1}\right] + \frac{1}{k+1} - \frac{1}{k+2}$

But $\sum_{r=1}^{k} \frac{1}{r} - \frac{1}{r+1} = \frac{k}{k+1}$ *from Assumption step*

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{k}{k+1} + \frac{1}{k+1} - \frac{1}{k+2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{k(k+2)+1(k+2)-1(k+1)}{(k+1)(k+2)}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{k^2+2k+1}{(k+1)(k+2)}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{(k+1)^2}{(k+1)(k+2)}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{k+1}{k+2}$

$\Rightarrow \sum_{r=1}^{k+1} \frac{1}{r} - \frac{1}{r+1} = \frac{k+1}{(k+1)+1}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k + 1$

Example 4

Prove by mathematical induction that

$$\sum_{r=1}^{n} (r-1) \cdot (r-1)! = n! - 1$$

Solution

Initial Step

When $n = 1$

$LHS = (1 - 1) \cdot (1 - 1)! = 0$

$RHS = 1! - 1 = 0$

$\therefore LHS = RHS$ *the statement is true for n=1*

When $n = 2$

.

$LHS = [(1 - 1) \cdot (1 - 1)!] + [(2 - 1) \cdot (2 - 1)!] = 0 \cdot 0! + 1 \cdot 1! = 1$

$RHS = 2! - 1 = 1$

$\therefore LHS = RHS$ *the statement is true for n=2*

<u>*When* $n = 3$</u>

$LHS = [(1-1)\cdot(1-1)!] + [(2-1)\cdot(2-1)!] + [(3-1)\cdot(3-1)!] = 0\cdot 0! + 1\cdot 1! + 2\cdot 2! = 5$

$RHS = n! - 1 = 6 - 1 = 5$

$\therefore LHS = RHS$ *the statement is true for n=3*

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\sum_{r=1}^{k}(r-1)\cdot(r-1)! = k! - 1$$

Inductive step

$\Rightarrow \sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$

$\Rightarrow \sum_{r=1}^{k+1}(r-1)\cdot(r-1)! = \left[\sum_{r=1}^{k}(r-1)\cdot(r-1)!\right] + (k+1-1)\cdot(k+1-1)!$

But $\sum_{r=1}^{k}(r-1)\cdot(r-1)! = k! - 1$ *from the assumption*

$\Rightarrow \sum_{r=1}^{k+1}(r-1)\cdot(r-1)! = (k!-1) + k\cdot k! = k\cdot k! + k! - 1$

$\Rightarrow \sum_{r=1}^{k+1}(r-1)\cdot(r-1)! = (k+1)\cdot k! - 1$

NB $\sim \begin{cases} (k+1)! & = (k+1)\cdot k\cdot(k-1)!\ldots 3\cdot 2\cdot 1 \\ & = (k+1)\cdot[k\cdot(k-1)!\ldots 3\cdot 2\cdot 1] \\ & = (k+1)\cdot k! \end{cases}$

$\Rightarrow \sum_{r=1}^{k+1}(r-1)\cdot(r-1)! = (k+1)! - 1$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k+1$

Example 5

Prove by induction that the sum of the first n terms in a geometric progression with first term a and common ratio r where $r > 1$ is given by $\frac{a(r^n-1)}{r-1}$

Solution

$f(i) = ar^{i-1}$ and thus we want to prove that $\sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$ for all $n \in \mathbb{N}$

Initial Step

When $n = 1$

$LHS = ar^{1-1} = a$

$RHS = \frac{a(r^1-1)}{r-1} = a$

$\therefore LHS = RHS$ *the statement is true for n=1*

When $n = 2$

.

$LHS = [ar^{1-1}] + [ar^{2-1}] = a + ar = a(r+1)$

$RHS = \frac{a(r^2-1)}{r-1} = \frac{a(r-1)(r+1)}{r-1} = a(r+1)$

$\therefore LHS = RHS$ *the statement is true for n=2*

When $n = 3$

$LHS = [ar^{1-1}] + [ar^{2-1}] + [ar^{3-1}] = a + ar + ar^2 = a(r^2+r+1)$

$RHS = \frac{a(r^3-1)}{r-1} = \frac{a(r-1)(r^2+r+1)}{r-1} = a(r^2+r+1)$

$\therefore LHS = RHS$ *the statement is true for n=3*

Assumption Step

Assume that the statement holds for n=k and thus we have

$$\sum_{i=1}^{k} ar^{i-1} = \frac{a(r^k-1)}{r-1}$$

Inductive step

When $n = k + 1$

$\Rightarrow \sum_{i=1}^{k+1} f(i) = \sum_{i=1}^{k} f(i) + f(k+1)$

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \left[\sum_{i=1}^{k} ar^{i-1}\right] + ar^{(k+1)-i}$

But $\sum_{i=1}^{k} ar^{i-1} = \frac{a(r^n-1)}{r-1}$ *from assumption.*

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \frac{a(r^k-1)}{r-1} + ar^k$

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \frac{a(r^k-1)+ar^k(r-1)}{r-1}$

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \frac{ar^k-a+ar^{k+1}-ar^k)}{r-1}$

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \frac{ar^{k+1}-a)}{r-1}$

$\Rightarrow \sum_{i=1}^{k+1} ar^{i-1} = \frac{a(r^{k+1}-1)}{r-1}$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k+1$

1.3 Products

Firstly , we need to understand the product notation. Product notation has 4 parts i.e the sign ($\prod$) , lower lim (below the sign) , upper limit (above the sign) and the function (just after sign) as shown below

$$\prod_{r=L.lim}^{U.lim} f(r)$$

For example $\prod_{r=1}^{4} r^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 = 576$

The general formula for $\prod_{r=1}^{n} r^2 = (n!)^2$.
$\Rightarrow \prod_{r=1}^{4} r^2 = (4!)^2 = 24^2 = 576$

Another example is $\prod_{r=1}^{7} 5 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 78\ 125$

The general formula of $\prod_{r=1}^{n} 5$ is 5^n.
$\Rightarrow \prod_{r=1}^{7} 5 = 5^7 = 78\ 125$

Example 1

Prove by induction that $\prod_{r=1}^{n} 5 = 5^n$

Solution

Initial Step

When $n = 1$

$LHS = 5$

$RHS = 5^1 = 5$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

When $n = 2$

.

$LHS = 5 \cdot 5 = 25$

$RHS = 5^2 = 25$

$\therefore LHS = RHS$ *the statement is true for* $n = 2$

When $n = 3$

$LHS = 5 \cdot 5 \cdot 5 = 125$

$RHS = 5^3 = 125$

$\therefore LHS = RHS$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\prod_{r=1}^{k} 5 = 5^k$$

Inductive step

When $n = k + 1$

$\Rightarrow \prod_{r=1}^{k+1} 5 = \left[\prod_{r=1}^{k} 5\right] \times 5$

But $\prod_{r=1}^{k} 5 = 5^k$ *from assumption*

$\Rightarrow \prod_{r=1}^{k+1} 5 = 5^k \times 5$

$\Rightarrow \prod_{r=1}^{k+1} 5 = 5^k \times 5^1 = 5^{k+1}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k + 1$

NB $\sim$ When we are using the method of mathematical induction to prove a product formula , we use the following identity in the inductive step

Notes

★ Given that $\prod_{r=1}^{n} f(r) = g(n)$,then $\prod_{r=1}^{k+1} f(r)$ (in the inductive step) is found using the following identity

$$\prod_{r=1}^{k+1} f(r) = \left[\prod_{r=1}^{k} f(r)\right] \times f(k+1)$$

and the result should be the same as the result we get by direct substitution i.e g(k+1) but direct substitution should not be used in the inductive step.

Proof

$$\begin{aligned}\Rightarrow \prod_{r=1}^{k+1} f(r) &= f(1) \times f(2) \times f(3) \times \ldots \times f(k-1) \times f(k) \times f(k+1) \\ &= [f(1) \times f(2) \times f(3) \times \ldots \times f(k-1) \times f(k)] \times f(k+1) \\ &= \left[\prod_{r=1}^{k} f(r)\right] \times f(k+1)\end{aligned}$$

Example 2

Prove by induction that the product of the first n terms in a geometric progression with first term a and common ratio r where $r > 1$ is given by $a^n r^{\left(\frac{n(n-1)}{2}\right)}$

Solution

$f(i) = ar^{i-1}$ and therefore we want to prove that $\prod_{i=1}^{n} ar^{i-1} = a^n r^{(\frac{n(n-1)}{2})}$

Initial Step

When $n = 1$

$LHS = ar^{1-1} = a$

$RHS = a^1 r^{(\frac{1(1-1)}{2})} = ar^0 = a$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

When $n = 2$

.

$LHS = [ar^{1-1}] \cdot [ar^{2-1}] = a^2 r$

$RHS = a^2 r^{(\frac{2(2-1)}{2})} = a^2 r^1 = a^2 r$

$\therefore LHS = RHS$ *the statement is true for* $n = 2$

When $n = 3$

$LHS = [ar^{1-1}] \cdot [ar^{2-1}] \cdot [ar^{3-1}] = a \cdot ar \cdot ar^2 = a^3 r^3$

$RHS = a^3 r^{(\frac{3(3-1)}{2})} = a^3 r^3$

$\therefore LHS = RHS$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\prod_{i=1}^{k} ar^{i-1} = a^k r^{(\frac{k(k-1)}{2})}$$

Inductive step

When $n = k + 1$

$\Rightarrow \prod_{i=1}^{k+1} f(i) = [\prod_{i=1}^{k} f(i)] \times f(k+1)$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = \left[\prod_{i=1}^{k} ar^{i-1}\right] \times ar^{(k+1)-1}$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = \left[\prod_{i=1}^{k} ar^{i-1}\right] \times ar^{k}$

But $\prod_{i=1}^{k} ar^{i-1} = a^k r^{(\frac{k(k-1)}{2})}$ *from assumption.*

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^k r^{(\frac{k(k-1)}{2})} \times ar^k$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^k \cdot a^1 \cdot r^{(\frac{k(k-1)}{2})} \times r^k$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^{k+1} \cdot r^{(\frac{k(k-1)}{2}+k)}$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^{k+1} \cdot r^{(\frac{k^2-k+2k}{2})}$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^{k+1} \cdot r^{(\frac{(k+1)k}{2})}$

$\Rightarrow \prod_{i=1}^{k+1} ar^{i-1} = a^{k+1} \cdot r^{(\frac{(k+1)(k+1-1)}{2})}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k + 1$

Example 3

Prove by induction that

$$\prod_{r=1}^{n} cos2^r x = \frac{sin2^{n+1}x}{2^n sin2x}$$

Solution

Initial Step

When $n = 1$

$LHS = cos2^1 x = cos2x$

$RHS = \frac{sin2^{1+1}x}{2^1 sin2x} = \frac{sin4x}{2sin2x} = \frac{2sin2xcos2x}{2sin2x} = cos2x$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

When $n = 2$

.

$LHS = cos2x \cdot cos2^2 x = cos2xcos4x$

$RHS = \frac{sin2^{2+1}x}{2^2 sin2x} = \frac{sin8x}{4sin2x} = \frac{2sin4xcos4x}{4sin2x} = \frac{2(2sin2xcos2x)cos4x}{4sin2x} = \frac{4sin2xcos2xcos4x}{4sin2x} = cos2xcos4x$

$\therefore LHS = RHS$ *the statement is true for* $n = 2$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\prod_{r=1}^{k} cos2^r x = \frac{sin2^{k+1}x}{2^k sin2x}$$

Inductive step

When $n = k + 1$

$\Rightarrow \prod_{i=1}^{k+1} f(i) = [\prod_{i=1}^{k} f(i)] \times f(k+1)$

$\Rightarrow \prod_{i=1}^{k+1} cos2^r x = \left[\prod_{i=1}^{k} cos2^r x\right] \times cos2^{k+1}x$

But $\prod_{r=1}^{k} cos2^r x = \frac{sin2^{k+1}x}{2^k sin2x}$ *from assumption.*

$\Rightarrow \prod_{i=1}^{k+1} cos2^r x = \frac{sin2^{k+1}x}{2^k sin2x} \times cos2^{k+1}x$

$\Rightarrow \prod_{i=1}^{k+1} cos2^r x = \frac{sin2^{k+1}xcos2^{k+1}x}{2^k sin2x}$

NB$\sim$ *Using the identity* $sin2A = 2sinAcosA$,

$\Rightarrow sinAcosA = \frac{sin2A}{2}$

$If\ A = 2^{k+1}x,$

$\Rightarrow sin2^{k+1}xcos2^{k+1}x = \frac{sin(2 \cdot 2^{k+1}x)}{2}$

$\Rightarrow \prod_{i=1}^{k+1} cos2^r x = \frac{sin(2 \cdot 2^{k+1}x)}{2 \cdot 2^k sin2x}$

$\Rightarrow \prod_{i=1}^{k+1} cos2^r x = \frac{sin(2^{(k+1)+1}x)}{2^{k+1} sin2x}$

NB $\sim \frac{sin(2^{(k+1)+1}x)}{2^{k+1} sin2x}$ *is the same as* $g(k+1)$ *where* $g(n) = \frac{sin2^{n+1}x}{2^n sin2x}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k + 1$

Example 4

Given that $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$. Prove by mathematical induction that $A^{n+1} = \begin{pmatrix} 9^n & 4\cdot 9^n \\ 2\cdot 9^n & 8\cdot 9^n \end{pmatrix}$

Solution

NB $\sim A^{n+1} = \prod_{r=1}^{n+1} A$

Initial Step

When $n = 1$

$LHS = A^{1+1} = A^2 = A \times A$

$$LHS = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 1\times 1 + 4\times 2 & 1\times 4 + 4\times 8 \\ 2\times 1 + 8\times 2 & 2\times 4 + 8\times 8 \end{pmatrix} = \begin{pmatrix} 9 & 36 \\ 18 & 72 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 9^1 & 4\cdot 9^1 \\ 2\cdot 9^1 & 8\cdot 9^1 \end{pmatrix} = \begin{pmatrix} 9 & 36 \\ 18 & 72 \end{pmatrix}$$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$A^{k+1} = \begin{pmatrix} 9^k & 4\cdot 9^k \\ 2\cdot 9^k & 8\cdot 9^k \end{pmatrix}$$

Inductive step

When $n = k + 1$

$\Rightarrow LHS = A^{(k+1)+1} = A \times A^{k+1}$

But $A^{k+1} = \begin{pmatrix} 9^k & 4\cdot 9^k \\ 2\cdot 9^k & 8\cdot 9^k \end{pmatrix}$ *from assumption*

$$\Rightarrow LHS = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 9^k & 4\cdot 9^k \\ 2\cdot 9^k & 8\cdot 9^k \end{pmatrix}$$

$$\Rightarrow LHS = \begin{pmatrix} 9^k + 4\times 2\cdot 9^k & 4\cdot 9^k + 4\times 8\cdot 9^k \\ 2\times 9^k + 8\times 2\cdot 9^k & 2\times 4\cdot 9^k + 8\times 8\cdot 9^k \end{pmatrix}$$

$$\Rightarrow LHS = \begin{pmatrix} 9 \cdot 9^k & 36 \cdot 9^k \\ 18 \cdot 9^k & 72 \cdot 9^k \end{pmatrix}$$

$$\Rightarrow LHS = \begin{pmatrix} 9^1 \cdot 9^k & 4 \cdot 9^1 \cdot 9^k \\ 2 \cdot 9^1 \cdot 9^k & 8 \cdot 9^1 \cdot 9^k \end{pmatrix}$$

$$\Rightarrow LHS = \begin{pmatrix} 9^{k+1} & 4 \cdot 9^{k+1} \\ 2 \cdot 9^{k+1} & 8 \cdot 9^{k+1} \end{pmatrix}$$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$.

EXERCISE 1

1. Use the principle of mathematical induction to prove that the following statements are true for all $n \in \mathbb{N}$
   (a) $\sum_{r=1}^{n} 2r = n(n+1)$
   (b) $\sum_{r=1}^{n} 3r^2 + r = n(n+1)^2$
   (c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$
   (d) $\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$
   (e) $\sum_{r=1}^{n} 3r(r+1) = n(n+1)(n+2)$
   (f) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$
   (g) $\sum_{r=1}^{n} 6 \cdot 7^r = 7(7^n - 1)$
   (h) $\sum_{r=1}^{n} \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$
   (i) $\sum_{r=1}^{n} cos(2r-1)x = \frac{sin2nx}{2sinx}$
   (j) $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \ldots + (3n-2) \cdot (3n+1) = 3n^3 + 3n^2 - 2n$

2. Prove the following statement by using the method of mathematical induction

$$3 \cdot 1! + 7 \cdot 2! + 13 \cdot 3! + \ldots + (n^2 + n + 1) \cdot n! = (n+1)^2 \cdot n! - 1$$

3. Prove by induction that $\prod_{r=1}^{n} x^{2r} = x^{n(n+1)}$

4. Prove by using the principle of mathematical induction that

$$\prod_{j=1}^{n} j^m = (n!)^m$$

where $m \ , \ n \ \in \mathbb{Z}^+$

5. Prove by induction that $\prod_{r=1}^{n} A = \begin{pmatrix} 2^{n-1} & 0 & 2^{n-1} \\ 0 & 1 & 0 \\ 2^{n-1} & 0 & 2^{n-1} \end{pmatrix}$ where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

1.4 Derivatives

In this section , we should know how to differentiate different types of functions. Mathematical induction is used to prove the general formula for the nth derivative of a certain function or for the derivative for a certain function. We are going to start with nth derivatives.

1.4.1 nth derivatives

If we are asked to find $\frac{d^3(x^{12})}{dx^3}$, we differentiate $y = x^{12}$ three times i.e

$y = x^{12}$

$\frac{dy}{dx} = 12x^{11}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 132x^{10}$

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 1320x^9$

The general formula for the nth derivative of $y = x^r$ is given by $\frac{d^ny}{dx^n} = \frac{r!x^{r-n}}{(r-n)!}$ for $r \geq n$. So when $r = 12$ and $n = 3$, then we have $\frac{d^3y}{dx^3} = \frac{12!x^{12-3}}{(12-3)!} = \frac{12!x^9}{9!} = 1320x^9$.

In the example above , you can see that its easy and less time consuming to compute the nth derivative using the general formula . So we use mathematical induction to prove if the general formula for the nth derivative is true for all $n \in \mathbb{N}$. The following example show the proof of the above formula using the principle of mathematical induction.

Example 1

Prove by induction that $\frac{d^n x^r}{dx^n} = \frac{r! x^{r-n}}{(r-n)!}$

Solution

Initial Step

When $n = 1$

$LHS = \frac{d}{dx}(x^r) = rx^{r-1}$

$RHS = \frac{r! x^{r-1}}{(r-1)!} = \frac{r \cdot (r-1)! x^{r-1}}{(r-1)!} = rx^{r-1}$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

When $n = 2$

.

$LHS = \frac{d}{dx}(rx^{r-1}) = r(r-1)x^{r-2}$

$RHS = \frac{r! x^{r-2}}{(r-2)!} = \frac{r \cdot (r-1) \cdot (r-2) \cdot (r-3) \cdots 3 \cdot 2 \cdot 1 \times x^{r-2}}{(r-2) \cdot (r-3) \cdots 3 \cdot 2 \cdot 1} = r(r-1)x^{r-2}$

$\therefore LHS = RHS$ *the statement is true for* $n = 2$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d^k x^r}{dx^k} = \frac{r! x^{r-k}}{(r-k)!}$$

Inductive step

When $n = k + 1$

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

But $\frac{d^k x^r}{dx^k} = \frac{r! x^{r-k}}{(r-k)!}$ *from assumption*

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{r! x^{r-k}}{(r-k)!}\right)$

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{(r-k) r! x^{r-k-1}}{(r-k)!}$

$\Rightarrow \frac{d^{k+1} y}{dx^{k+1}} = \frac{(r-k) r! x^{r-k-1}}{(r-k) \cdot (r-k-1)!}$

NB $\sim (r-k)! = (r-k) \cdot (r-k-1) \cdot (r-k-2) \ldots 3 \cdot 2 \cdot 1$,
but $(r-k-1) \cdot (r-k-2) \ldots 3 \cdot 2 \cdot 1 = (r-k-1)!$

$\Rightarrow (r-k)! = (r-k)\cdot(r-k-1)!$

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = \frac{r!x^{r-(k+1)}}{(r-(k+1))!}$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2$, . . . , for $n = k$ and for $n = k+1$

Notes

★ The identity which is used in the inductive step for the nth derivatives is given by

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

Example 2

Prove by induction that $\frac{d^n y}{dx^n} = sin\left(x + \frac{(n+1)\pi}{2}\right)$ where $y = cosx$

Solution

Initial Step

When $n = 1$

$LHS = \frac{d\ (cosx)}{dx} = -sinx$

$RHS = sin\left(x + \frac{(1+1)\pi}{2}\right) = sin(x+\pi) = sinxcos\pi + cosxsin\pi = sinx(-1) + cosx(0) = -sinx$

NB $\sim$ $sin(A+B) = sinAcosB + cosAsinB$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

When $n = 2$

.

$LHS = \frac{d^2cosx}{dx^2} = \frac{d}{dx}(-sinx) = -cosx$

$RHS = sin\left(x + \frac{(2+1)\pi}{2}\right) = sin\left(x + \frac{3\pi}{2}\right) = sinxcos\left(\frac{3\pi}{2}\right) + cosxsin\left(\frac{3\pi}{2}\right) = sinx(0) + cosx(-1) = -cosx$

$\therefore LHS = RHS$ *the statement is true for* $n = 2$

<u>When $n = 3$</u>

$LHS = \frac{d^3 cosx}{dx^3} = \frac{d}{dx}(-cosx) = sinx$

$RHS = sin\left(x + \frac{(3+1)\pi}{2}\right) = sin(x + 2\pi) = sinxcos2\pi + cosxsin2\pi = sinx(1) + cosx(0) = sinx$

$\therefore LHS = RHS$ *the statement is true for* $n = 3$

<u>Assumption Step</u>

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d^k y}{dx^k} = sin\left(x + \frac{(k+1)\pi}{2}\right)$$

<u>Inductive step</u>

When $n = k + 1$

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

But $\frac{d^k y}{dx^k} = sin\left(x + \frac{(k+1)\pi}{2}\right)$ *from assumption*

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(sin\left(x + \frac{(k+1)\pi}{2}\right)\right)$

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = 1cos\left(x + \frac{(k+1)\pi}{2}\right)$

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = sin\left[\left(x + \frac{(k+1)\pi}{2}\right) + \frac{\pi}{2}\right]$ *Using the identity* $cosX = sin(X + \frac{\pi}{2})$

$\Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = sin\left(x + \frac{[(k+1)+1]\pi}{2}\right)$

Therefore , the statement is true for $n = k + 1$

NB $\sim$ $sin\left(x + \frac{[(k+1)+1]\pi}{2}\right)$ *is the same as* $g(k+1)$ *given* $g(n) = sin\left(x + \frac{(n+1)\pi}{2}\right)$

<u>Conclusion</u>

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k + 1$

Now lets look at the following examples (for the first derivatives).

1.4.2 1st derivatives

Definition of a derivative : Given $f(x)$ where $f(x)$ is any function (continuous), then $f'(x)$ is given by

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For example , if $f(x) = x^n$, then

$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$f'(x) = \lim_{h\to 0} \frac{x^n + hnx^{n-1} + \binom{n}{2} h^2 x^{n-2} + \binom{n}{3} h^3 x^{n-3} + \ldots + h^n - x^n}{h}$

$f'(x) = \lim_{h\to 0} \frac{hnx^{n-1} + \binom{n}{2} h^2 x^{n-2} + \binom{n}{3} h^3 x^{n-3} + \ldots + h^n}{h}$

$f'(x) = \lim_{h\to 0} \left[nx^{n-1} + \binom{n}{2} hx^{n-2} + \binom{n}{3} h^2 x^{n-3} + \ldots + h^{n-1} \right]$

$f'(x) = nx^{n-1} + \binom{n}{2} (0) x^{n-2} + \binom{n}{3} (0)^2 x^{n-3} + \ldots + (0)^{n-1}$

$f'(x) = nx^{n-1}$

We derive the general formula for the derivative of $f(x) = x^n$. So we use mathematical induction to prove that the formula is true for all $n \in \mathbb{N}$ as shown in the example below.

NB $\sim$ the LHS of the initial step is differentiated using differentiation from first principles (i.e using the formula of $f'(x)$) ,and we use product rule in the inductive step

Example 3

Prove by induction that $\frac{d}{dx}(x^n) = nx^{n-1}$

Solution

Initial Step

When $n = 1$

$LHS = \frac{d}{dx}(x^1)$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{(x+h)-x}{h}$

$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = \mathbf{1}$

$RHS = 1x^{1-1} = 1x^0 = \mathbf{1}$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

Inductive step

When $n = k + 1$

$\Rightarrow LHS = \frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}(x^k \cdot x)$

$\Rightarrow LHS = x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k)$ (using product rule)

$\Rightarrow LHS = x^k \cdot 1 + x \cdot \frac{d}{dx}(x^k)$ since $\frac{d}{dx}(x) = 1$ {already proved - initial step}

But $\frac{d}{dx}(x^k) = kx^{k-1}$ *from assumption*

$\Rightarrow LHS = x^k + x(kx^{k-1}) = x^k + kx^k$

$\Rightarrow LHS = (k+1)x^k$

$\Rightarrow LHS = (k+1)x^{(k+1)-1}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k + 1$

Example 4

Prove by induction that $\frac{d}{dx}sin(nx) = ncos(nx)$

Solution

Initial Step

When $n = 1$

$LHS \ = \frac{d}{dx}sinx = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{sin(x+h)-sinx}{h} = \lim_{h\to 0}\frac{sinxcosh+cosxsinh-sinx}{h}$

$LHS \ = \lim_{h\to 0}\frac{sinxcosh-sinx}{h} + \lim_{h\to 0}\frac{cosxsinh}{h} = sinx\lim_{h\to 0}\frac{cosh-1}{h} + cosx\lim_{h\to 0}\frac{sinh}{h}$

In this section we are not concerned with evaluating limits. So we are going to use a very simple method i.e h is approaching zero , so we let h be a very small number , for example $h = 0,000\ 000\ 001$ .Thus we have $\lim_{h\to 0}\frac{cosh-1}{h} = 0$ and $\lim_{h\to 0}\frac{sinh}{h} = 1$

$LHS \ = sinx(0) + cosx(1) = cosx$

$RHS = 1cos(1x) = cosx$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d}{dx}sin(kx) = kcos(kx)$$

Inductive step

When $n = k + 1$

$\Rightarrow LHS \ = \frac{d}{dx}sin[(k+1)x]$

$\Rightarrow LHS \ = \frac{d}{dx}sin(kx + x)$

$\Rightarrow LHS \ = \frac{d}{dx}(sinkxcosx + coskxsinx)$

$\Rightarrow LHS \ = sinkx(-sinx) + cosx\frac{d}{dx}(sinkx) + cosxcoskx - ksinxsinkx$

$\Rightarrow LHS \ = cosx\frac{d}{dx}(sinkx) + cosxcoskx - [ksinxsinkx + sinxsinkx]$

But $\frac{d}{dx}sin(kx) = kcos(kx)$ *from assumption*

$\Rightarrow LHS \ = kcosxcoskx + cosxcoskx - [ksinxsinkx + sinxsinkx]$

$\Rightarrow LHS \ = (k+1)cosxcoskx - (k+1)sinxsinkx$

$\Rightarrow LHS \ = (k+1)[cosxcoskx - sinxsinkx]$

$\Rightarrow LHS \ = (k+1)cos(kx+x)$

$\Rightarrow LHS \ = (k+1)cos(k+1)x$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

Example 5

Prove by induction that $\frac{d}{dx}(x^n e^{nx}) = n(x+1)x^{n-1}e^{nx}$

Solution

Initial Step

When $n = 1$

★$LHS = \frac{d}{dx}(x^1 e^{1x}) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)e^{(x+h)}-xe^x}{h}$

$LHS \ = e^x \lim_{h\to 0} \frac{(x+h)e^h - x}{h} = e^x \lim_{h\to 0} \frac{xe^h + he^h - x}{h}$

$LHS \ = e^x \left[\lim_{h\to 0} \frac{xe^h - x}{h} + \lim_{h\to 0} \frac{he^h}{h}\right]$

$LHS \ = e^x \left[x \lim_{h\to 0} \frac{e^h - 1}{h} + \lim_{h\to 0} e^h\right]$

$LHS \ = e^x[x(1) + 1]$

$LHS \ = (x+1)e^x$

★$RHS = 1(x+1)x^{1-1}e^{1x} = (x+1)e^x$

$\therefore LHS = RHS$ *the statement is true for* $n = 1$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d}{dx}(x^k e^{kx}) = k(x+1)x^{k-1}e^{kx}$$

Inductive step

When $n = k + 1$

$\Rightarrow LHS \; = \frac{d}{dx}(x^{k+1}e^{(k+1)x})$

$\Rightarrow LHS \; = \frac{d}{dx}(xe^x \cdot x^k e^{kx})$

Using product rule

$\Rightarrow LHS \; = (x+1)e^x \cdot x^k e^{kx} + xe^x \cdot \frac{d}{dx}(x^k e^{kx})$

But $\frac{d}{dx}(x^k e^{kx}) = k(x+1)x^{k-1}e^{kx}$ *from assumption*

$\Rightarrow LHS \; = (x+1)e^x \cdot x^k e^{kx} + xe^x \cdot k(x+1)x^{k-1}e^{kx}$

$\Rightarrow LHS \; = 1(x+1)x^k e^{kx+x} + k(x+1)x^k e^{kx+x}$

$\Rightarrow LHS \; = (k+1)(x+1)x^k e^{kx+x}$

$\Rightarrow LHS \; = (k+1)(x+1)x^{(k+1)-1}e^{(k+1)x}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

Example 6

Prove by induction that

$$\frac{d^n}{dx^n}\left(\frac{1}{2x+1}\right) = \frac{(-2)^n \cdot n!}{(2x+1)^{n+1}}$$

Solution

Initial Step

When $n = 1$

$LHS = \frac{d^1}{dx^1}\left(\frac{1}{2x+1}\right) = \frac{d}{dx}\left(\frac{1}{2x+1}\right) = \frac{d}{dx}(2x+1)^{-1} = -1(2x+1)^{-2} \times 2 = \frac{-2}{(2x+1)^2}$ Note 1

$RHS = \frac{(-2)^1 \cdot 1!}{(2x+1)^{1+1}} = \frac{-2 \cdot 1}{(2x+1)^2} = \frac{-2}{(2x+1)^2}$

$\therefore LHS = RHS$ (*the statement is true for n=1*)

Inductive hypothesis

Assume that the statement holds for $n = k$ and thus we have

$$\frac{d^k}{dx^k}\left(\frac{1}{2x+1}\right) = \frac{(-2)^k \cdot k!}{(2x+1)^{k+1}}$$

Inductive step Note 2

When $n = k + 1$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+1}\right) = \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\frac{1}{2x+1}\right)\right]$

but $\frac{d^k}{dx^k}\left(\frac{1}{2x+1}\right) = \frac{(-2)^k \cdot k!}{(2x+1)^{k+1}}$ *from inductive hypothesis.*

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+1}\right) = \frac{d}{dx}\left[\frac{(-2)^k \cdot k!}{(2x+1)^{k+1}}\right]$

$= \frac{d}{dx}\left[(-2)^k \cdot k! \cdot (2x+1)^{-(k+1)}\right]$

$= (-2)^k \cdot k! \cdot [-(k+1)](2x+1)^{-(k+1)-1} \times 2$ *Note 1*

$= (-2) \cdot (-2)^k \cdot k! \cdot (k+1)(2x+1)^{-(k+2)}$

$= (-2)^{k+1}(k+1) \cdot k!(2x+1)^{-(k+2)}$

$= \frac{(-2)^{k+1}(k+1) \cdot k!}{(2x+1)^{k+2}}$

$= \frac{(-2)^{k+1}(k+1)!}{(2x+1)^{k+2}}$ *Note 3*

$= \frac{(-2)^{k+1}(k+1)!}{(2x+1)^{k+1+1}}$ $=$ *Note 2*

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

NOTES

1. $\frac{d}{dx}[a(bx+c)^n] = an(bx+c)^{n-1} \times \frac{d}{dx}(bx+c) = an(bx+c)^{n-1} \times b$

2. When $n = k+1$, by direct substitution, we have

   $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+1}\right) = \frac{(-2)^{k+1}\cdot(k+1)!}{(2x+1)^{k+1+1}}$.
   Note : we should not use direct substitution in the inductive step.

3. $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$

   When $n = k$, $k! = k \times (k-1) \times (k-2) \times (k-3) \times \ldots \times 3 \times 2 \times 1$

   When $n = k+1$, $(k+1)! = (k+1) \times k \times (k-1) \times (k-2) \times (k-3) \times \ldots \times 3 \times 2 \times 1$

   Therefore $(k+1)! = (k+1) \times k!$

   For example,
   $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   &
   $5! = 5 \times 4! = 5 \times (4 \times 3 \times 2 \times 1) = 5 \times 24 = 120$

1.5 Divisions

In this section , we need to understand the term **divisible**. If a statement $P(n)$ is divisible by N where n and N is are positive integers (natural numbers) , then $P(n) \div N$ is an integer . Let that integer be **a**.

$\gg$ $\frac{P(n)}{N} = a$, *which implies that* $P(n) = Na$

So if we are given a question "Prove by induction that $P(n)$ is divisible by N" , the first thing to do is to equate $P(n)$ to Na and then prove by induction that $P(n) = Na$ for some $a \in \mathbb{Z}$

Example 1

Prove by induction that $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$.

Solution

We are asked to prove that $11^n - 1 = 10a$ *for some integer* a

Initial Step

When $n = 1$
$LHS = 11^1 - 1 = 10 = 10(1) = RHS$
$\therefore$ *the statement is true for* $n = 1$

When $n = 2$
. $LHS = 11^2 - 1 = 120 = 10(12) = RHS$
$\therefore$ *the statement is true for* $n = 2$

When $n = 3$
$LHS = 11^3 - 1 = 1\ 330 = 10(133) = RHS$
$\therefore$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$11^k - 1 = 10a$$

Inductive step

When $n = k + 1$

$\Rightarrow P(k+1) = 11^{k+1} - 1 = 11^k \cdot 11^1 - 1$

$\Rightarrow P(k+1) = 11 \cdot 11^k - 1$

But $11^k = 10a + 1$ *from assumption*

$\Rightarrow P(k+1) = 11(10a+1) - 1 = 11 \cdot 10a + 11 - 1$

$\Rightarrow P(k+1) = 10 \cdot 11a + 10$ *NB ∼ multiplication is commutative i.e* $11 \times 10 = 10 \times 11$

$\Rightarrow \underline{\underline{P(k+1) = 10[11a+1]}}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k + 1$

Example 2

Prove by induction that $n^5 - n$ is divisible by 5

Solution

Initial Step

When $n = 1$
$LHS = 1^5 - 1 = 0 = 5(0)$
$\therefore$ *the statement is true for* $n = 1$

When $n = 2$
. $LHS = 2^5 - 2 = 30 = 5(6)$
$\therefore$ *the statement is true for* $n = 2$

When $n = 3$
$LHS = 3^5 - 3 = 240 = 5(48)$
$\therefore$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$k^5 - k = 5a$$

Inductive step

When $n = k + 1$

$\Rightarrow P(k + 1) = (k + 1)^5 - (k + 1)$

$\Rightarrow P(k + 1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k + 1)$

$\Rightarrow P(k + 1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$

$\Rightarrow P(k + 1) = (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$

But $k^5 - k = 5a$ *from assumption*

$\Rightarrow P(k+1) = 5a + 5k^4 + 10k^3 + 10k^2 + 5k$

$\Rightarrow \underline{\underline{P(k+1) = 5[a + k^4 + 2k^3 + 2k^2 + k]}}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k + 1$

Example 3

Prove by induction that $n^3 - n$ is divisible by 6

Solution

Initial Step

When $n = 1$
$LHS = 1^3 - 1 = 0 = 6(0)$
$\therefore$ *the statement is true for* $n = 1$

When $n = 2$
$LHS = 2^3 - 2 = 6 = 6(1)$
$\therefore$ *the statement is true for* $n = 2$

When $n = 3$
$LHS = 3^3 - 3 = 24 = 6(4)$
$\therefore$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$k^3 - k = 6a$$

Inductive step

When $n = k + 1$

$\Rightarrow P(k+1) = (k+1)^3 - (k+1)$

$\Rightarrow P(k+1) = k^3 + 3k^2 + 3k + 1 - (k-1)$

$\Rightarrow P(k+1) = k^3 + 3k^2 + 3k - k$

$\Rightarrow P(k+1) = (k^3 - k) + 3(k^2 + k)$

But $k^3 - k = 6a$ *from assumption*

$\Rightarrow P(k+1) = 6a + 3(k^2 + k)$

$\sim k^2 + k$ *is even for all* $k \in \mathbb{N}$

Proof

Using proof by cases

When k is even i.e $k = 2m$ **for some** $m \in \mathbb{Z}^+$
then $k^2 + k = (2m)^2 + 2m = 4m^2 + 2m = 2[2m^2 + m]$ *thus is divisible by 2 .*

When k is odd i.e $k = 2m - 1$ **for some** $m \in \mathbb{Z}^+$
then $k^2 + k = (2m-1)^2 + (2m-1) = 4m^2 - 4m + 1 + 2m - 1 = 4m^2 - 2m = 2[m^2 - m]$ *thus is divisible by 2 .*

So this implies that $k^2 + k = 2b$ for some $b \in \mathbb{Z}^+$

$\Rightarrow P(k+1) = 6a + 3(2b) = 6a + 6b = 6[a+b]$

$\Rightarrow \underline{\underline{P(k+1) = 6[a+b]}}$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k+1$

Example 4

Prove by induction that $6^{2n-1} + 8^{2n-1}$ is divisible by 14

Solution

Initial Step

When $n = 1$
$LHS = 6^{2(1)-1} + 8^{2(1)-1} = 6^1 + 8^1 = 14 = 14(1)$
$\therefore$ *the statement is true for* $n = 1$

When $n = 2$
. $LHS = 6^{2(2)-1} + 8^{2(2)-1} = 6^3 + 8^3 = 728 = 14(52)$
$\therefore$ *the statement is true for* $n = 2$

When $n = 3$
$LHS = 6^{2(3)-1} + 8^{2(3)-1} = 6^5 + 8^5 = 40\ 544 = 14(2\ 896)$
$\therefore$ *the statement is true for* $n = 3$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$6^{2k-1} + 8^{2k-1} = 14a$$

Inductive step

When $n = k + 1$

$\Rightarrow P(k+1) = 6^{2(k+1)-1} + 8^{2(k+1)-1}$

$\Rightarrow P(k+1) = 6^{2k+2-1} + 8^{2k+2-1}$

$\Rightarrow P(k+1) = 6^{2k-1+2} + 8^{2k-1+2}$

$\Rightarrow P(k+1) = 6^{2k-1} \cdot 6^2 + 8^{2k-1} \cdot 8^2$

$\Rightarrow P(k+1) = 36 \cdot 6^{2k-1} + 64 \cdot 8^{2k-1}$

But $6^{2k-1} + 8^{2k-1} = 14a$ *from assumption*

NB $\sim$ *We can make* 6^{2k-1} *or* 8^{2k-1} *the subject of formula*

$\gg 6^{2k-1} = 14a - 8^{2k-1}$

$\Rightarrow P(k+1) = 36 \cdot (14a - 8^{2k-1}) + 64 \cdot 8^{2k-1}$

$\Rightarrow P(k+1) = 36 \cdot 14a - 36 \cdot 8^{2k-1} + 64 \cdot 8^{2k-1}$

$\Rightarrow P(k+1) = 14 \cdot 36a + 28 \cdot 8^{2k-1}$ *NB* $\sim -36x + 64x = 28x$

$\Rightarrow \underline{\underline{P(k+1) = 14[36a + 2 \cdot 8^{2k-1}]}}$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$,. . . , for $n = k$ and for $n = k + 1$

Example 5

Prove by induction that $3^{n+1} \cdot 2^{6n+6} + 9 \cdot 3^{3n-1}$ is divisible by 15

Solution

Initial Step

When $n = 1$
$LHS = 3^{1+1} \cdot 2^{6(1)+6} + 9 \cdot 3^{3(1)-1} = 3^2 \cdot 2^{12} + 9 \cdot 3^2 = 36\ 945 = 15(2\ 463)$
$\therefore$ the statement holds for $n = 1$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$3^{k+1} \cdot 2^{6k+6} + 9 \cdot 3^{3k-1} = 15a$$

Inductive step

$\Rightarrow P(k+1) = 3^{(k+1)+1} \cdot 2^{6(k+1)+6} + 9 \cdot 3^{3(k+1)-1}$

$\Rightarrow P(k+1) = 3^{k+1+1} \cdot 2^{6k+6+6} + 9 \cdot 3^{3k-1+3}$

$\Rightarrow P(k+1) = 3^{k+1} \cdot 3^1 \cdot 2^{6k+6} \cdot 2^6 + 9 \cdot 3^{3k-1} \cdot 3^3$

$\Rightarrow P(k+1) = 192 \cdot 3^{k+1} \cdot 2^{6k+6} + 243 \cdot 3^{3k-1}$

But $3^{k+1} \cdot 2^{6k+6} = 15a - 9 \cdot 3^{3k-1}$ *from assumption*

$\Rightarrow P(k+1) = 192 \cdot (15a - 9 \cdot 3^{3k-1}) + 243 \cdot 3^{3k-1}$

$\Rightarrow P(k+1) = 192 \cdot 15a - 192 \cdot 9 \cdot 3^{3k-1} + 243 \cdot 3^{3k-1}$

$\Rightarrow P(k+1) = 15 \cdot 192a - 1\ 728 \cdot 3^{3k-1} + 243 \cdot 3^{3k-1}$

$\Rightarrow P(k+1) = 15 \cdot 192a - 1\ 485 \cdot 3^{3k-1}$

$\Rightarrow \underline{\underline{P(k+1) = 15[192a - 99 \cdot 3^{3k-1}]}}$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$ since the statement is true for $n = 1, 2, 3$, . . . , for $n = k$ and for $n = k+1$

EXERCISE 2

1. Prove the following statements by mathematical induction for all $n \in \mathbb{N}$
   (a) $n^2 + n$ is divisible by 2
   (b) $7^n - 3^n$ is divisible by 4
   (c) $2 \cdot 17^n - 2$ is divisible by 32
   (d) $3^{2n+1} - 3$ is divisible by 24
   (e) $12^n + 2 \cdot 3^{n+1}$ is divisible by 6
   (f) $n^4 + 10n^3 + 35n^2 + 50n + 24$ is divisible by 8
   (g) $3^{n+1} \cdot 2^{2n+2} + 2 \cdot 3^{n+2}$ is divisible by 18
2. Prove the following statements by mathematical induction for all $n \in \mathbb{N}$
   (a) $\frac{d^n(ln9x)}{dx^n} = \frac{(-1)^{n+1} \cdot (n-1)!}{x^n}$
   (b) $\frac{d^n(sinrx)}{dx^n} = r^2 sin(rx + \frac{n}{2}\pi)$
3. Prove by induction that $\frac{d}{dx}(3x^n + 5e^{nx}) = 3nx^{n-1} + 5ne^{nx}$ for all $n \in \mathbb{N}$
4. Prove by induction that $n^2 + 7n + 12$ is even for all $n \in \mathbb{N}$

1.6 Inequalities

In this section , we should know laws of inequalities . An inequality is used to compare two values . For example , if we are given that $a<b$, it means a is less than b , $a>b$ means a is greater than b , ...

There are many laws of inequalities and in this section the most important law is the **transitive law**. In inequalities , the transitive law states that :

Law 1 : Transitive law

If $a<b$ and $b<c$, then it implies that $a<c$ and the reverse is true

For example , if Victoria is younger than Peace and Peace is younger than Austin , then its obvious that Victoria is younger than Austin.

The following example is another example of transitive law

Another example of transitive law

If $a < b$ and $b = c$, then $a < c$

So if we are given $a<b = c = d<e = f<g<h = i< \ldots <y = z$,
then using the transitive law , it implies that $a<z$.

The following laws are also useful in this section.

Law 2

(a) If $b<c$, then $a + b < a + c$, and
(b) If $b>c$, then $a + b>a + c$

Law 3

(a) If b is positive , then $a + b>a$, and
(b) If b is negative , then $a + b<a$

Law 4

(a) If $a>0$ and $b>c$, then $ab>ac$
(b) If $a<0$ and $b>c$, then $ab<ac$

Law 5

If a and b are both positive or negative and $a<b$, then $\frac{1}{a}>\frac{1}{b}$ and the reverse is true

Example 1

Prove by induction that $n^2>n$ for $n>1$

Solution

Initial Step

NB $\sim$ *We start from* $n=2$ *since* $n>1$

When $n=2$

.

$LHS=2^2=4$

$RHS=2$

$\therefore LHS>RHS$, *the statement is true for* $n=2$

When $n=3$

$LHS=3^2=9$

$RHS=3$

$\therefore LHS>RHS$, *the statement is true for* $n=3$

Assumption Step

Assume that the statement holds for $n=k$ and thus we have

$$k^2 > k$$

Inductive step

NB ~ We want to prove that $(k+1)^2 > (k+1)$. So we start from $(k+1)^2$ to $k+1$

When $n = k + 1$

$\Rightarrow LHS = (k+1)^2$

$\Rightarrow LHS = k^2 + 2k + 1$

But $k^2 > k$ from the assumption

$\Rightarrow LHS \;=\; k^2 + 2k + 1 > k + 2k + 1$ (using Law 2(b))

$\Rightarrow LHS > k + 1 + 2k$ (transitive law)

$\Rightarrow LHS > k + 1 + 2k > k + 1$ (using Law 3(a) since 2k is positive [$k>1 => 2k>2$ & thus $2k>0$ since $2>0$])

$\Rightarrow LHS > k + 1$ (Using transitive law)

$\Rightarrow (k+1)^2 > (k+1)$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

Example 2

Prove by induction that $2^{n-1} < (n+1)!$

Solution

Initial Step

When $n = 1$

$LHS = 2^{1-1} = 2^0 = 1$

$RHS = (1+1)! = 2! = 2$

$\therefore LHS<RHS$, *the statement is true for* $n=1$

<u>*When* $n=2$</u>

.

$LHS = 2^{2-1} = 2^1 = 2$

$RHS = (2+1)! = 3! = 6$

$\therefore LHS<RHS$, *the statement is true for* $n=2$

<u>*When* $n=3$</u>

$LHS = 2^{3-1} = 2^2 = 4$

$RHS = (3+1)! = 4! = 24$

$\therefore LHS<RHS$, *the statement is true for* $n=3$

<u>Assumption Step</u>

Assume that the statement holds for $n=k$ and thus we have

$$2^{k-1}<(k+1)!$$

<u>Inductive step</u>

When $n=k+1$

NB ∼ We want to prove that $2^{k+1-1}<(k+1+1)!$. *So we start from* 2^{k+1-1} *to* $(k+1+1)!$

$\Rightarrow LHS = 2^{k+1-1} = 2^{k-1+1} = 2 \cdot 2^{k-1}$

$\Rightarrow LHS = 2 \cdot 2^{k-1}$

But $2^{k-1}<(k+1)!$ *from assumption*

$\Rightarrow LHS = 2 \cdot 2^{k-1}<2(k+1)!$ (Law 4(a))

$\Rightarrow LHS<2 \cdot (k+1)!$ (Using transitive law)

Since $k \in \mathbb{N}$,then $k>0$,which is the same as $0<k$. Adding 2 on both sides , we have $2<(k+2)$. Multiplying by (k+1)! both sides we have $2 \cdot (k+1)!<(k+2) \cdot (k+1)!$.

$\Rightarrow LHS < 2 \cdot (k+1)! < (k+2) \cdot (k+1)!$

$\Rightarrow LHS < (k+2) \cdot (k+1)!$ (Using transitive law)

$\Rightarrow LHS < (k+2) \cdot (k+1) \cdot k \cdot (k-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

$\Rightarrow LHS < (k+2)!$

$\Rightarrow LHS < (k+1+1)!$

$\Rightarrow 2^{k+1-1} < (k+1+1)!$

Therefore , the statement is true for $n = k+1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

Example 3

Prove by induction that $2^n \geq n^2$ for $n \geq 4$

Solution

Initial Step

When $n = 4$

$LHS = 2^4 = 16$

$RHS = 4^2 = 16$

$\therefore LHS \geq RHS \sim$ *the statement is true for* $n = 4$

When $n = 5$

$LHS = 2^5 = 32$

$RHS = 5^2 = 25$

$\therefore LHS \geq RHS \sim$ *the statement is true for* $n = 5$

Assumption Step

Assume that the statement holds for $n = k$ and thus we have

$$2^k \geq k^2$$

Inductive step

NB ~ We want to prove that $2^{k+1} \geq (k+1)^2$. So we start from 2^{k+1} to $(k+1)^2$

$\Rightarrow LHS = 2^{k+1} = 2 \cdot 2^k$

But $2^k \geq k^2$ from assumption

$\Rightarrow LHS = 2 \cdot 2^k \geq 2 \cdot k^2 = k^2 + k^2$

$\Rightarrow LHS \ \geq k^2 + k^2$

But we are given that $k \geq 4$ which implies that $k^2 \geq 4k$

$\Rightarrow LHS \ \geq k^2 + k^2 \geq k^2 + 4k$

$\Rightarrow LHS \ \geq k^2 + 2k + 2k$

But $k \geq 4$ which implies that $2k \geq 8$

$\Rightarrow LHS \ \geq k^2 + 2k + 2k \geq k^2 + 2k + 8$

$\Rightarrow LHS \ \geq k^2 + 2k + 8 \geq k^2 + 2k + 1$ (since $8 \geq 1$)

$\Rightarrow LHS \ \geq k^2 + 2k + 1 = (k+1)^2$

$\Rightarrow 2^{k+1} \geq (k+1)^2$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \geq 4$ where $n \in \mathbb{N}$ since the statement is true for $n = 4, 5$, . . . , for $n = k$ and for $n = k + 1$

Example 4

Prove by mathematical induction that $|sinnx| \leq n|sinx|$ for all $n \in \mathbb{N}$

Solution

Initial Step

When $n = 1$

$LHS = |sin1x| = |sinx|$

$RHS = 1|sinx| = |sinx|$

$\therefore LHS \leq RHS$ (*the statement is true for n=1*)

nb $\sim LHS \leq RHS$ *means* $LHS = RHS$ *or* $LHS<RHS$

When $n = 2$

$LHS = |sin2x| = |2sinxcosx| = 2|sinx||cosx|$

$RHS = 2|sinx|$

$\therefore LHS \leq RHS$ (*the statement is true for n=2*) *NOTE 1*

Inductive hypothesis

Assume that the statement holds for $n = k$ and thus we have

$$|sinkx| \leq k|sinx|$$

Inductive step

$When\ n = k + 1$ $\quad NOTE\ A$

$\Rightarrow LHS = |sin(k + 1)x|$

$= |sin(kx + x)|$ $\quad$ *by expanding* $(k + 1)x$

$= |sinkxcosx + coskxsinx|$ $\quad$ *using compound angle formula*

$\leq |sinkxcosx| + |coskxsinx|$ $\quad$ *using triangle inequality (NOTE 2)*

$\leq |sinkx||cosx| + |coskx||sinx|$ $\quad$ *using transitive law (NOTE 3)*

$But\ |sinkx| \leq k|sinx|$ $\quad$ *from inductive hypothesis*

$\Rightarrow LHS \quad \leq k|sinx||cosx| + |coskx||sinx|$

$But\ |coskx| \leq 1\ and\ |coskx| \leq 1$

$\Rightarrow k|sinx||cosx| + |coskx||sinx| \leq k|sinx| \cdot 1 + 1 \cdot |sinx|$ *using transitive law*

$\Rightarrow LHS \quad \leq k|sinx| + |sinx| = (k+1)|sinx|$

$\leq (k+1)|sinx| = RHS$

$\Rightarrow LHS \leq RHS$

$\Rightarrow |sin(k+1)x| \leq (k+1)|sinx|$

Therefore , the statement is true for $n = k + 1$

Conclusion

Therefore the statement is true for all $n \in \mathbb{N}$

NOTES FOR EXAMPLE 4

NOTE A

- *By direct substitution , when $n = k + 1$,then :*
  $|sin(k+1)| \leq (k+1)|sinx|$
- *So we must prove that $|sin(k+1)| \leq (k+1)|sinx|$ in the inductive step (do not use direct substitution) .*
- *We must apply inductive hypothesis in the inductive step.*
- *NB ∼ You should know Laws of Inequalities when proving mathematical induction statements with inequalities.*

NOTE 1

- *We know that cosine of any angle ranges from -1 to 1 i.e $-1 \leq cosx \leq 1$, which implies that $|cosx| \leq 1$.*
- *We also know that the modulus of any function is positive (or equal to 0) i.e $|sinx|$ is positive. If we multiply 2 positive numbers the result is positive (direct numbers) and thus $2|sinx|$ is positive .*

- *In Laws of inequalities , there is a Laws which states that :*
  If $a \leq b$ *and* $c \geq 0$ *,then* $ac \leq bc$
  This means that if we multiply both sides of an inequality by a positive number , then the inequality sign will not change.

- *Using the Law above,*
  $|cosx| \leq 1$ *and* $2|sinx| \geq 0$ *, then* $\underline{\underline{2|sinx||cosx| \leq 2|sinx|}}$ *, and therefore* $LHS \leq RHS$ *when* $n = 2$

NOTE 2

- *Triangle inequality states that*
  $|a + b| \leq |a| + |b|$

NOTE 3

- *Transitive law states that*
  If $a \leq b$ *and* $b \leq c$ *, then* $a \leq c$ & *If* $a \geq b$ *and* $b \geq c$ *, then* $a \geq c$

- *This implies that if* $a \leq b$ *and* $b = c$ *, then* $a \leq c$

- *Therefore if* $LHS \leq |sinkxcosx| + |coskxsinx|$ *and* $|sinkxcosx| + |coskxsinx| = |sinkx||cosx| + |cosk$
  , then $LHS \leq |sinkx||cosx| + |coskx||sinx|$

1.7 MATHEMATICAL INDUCTION QUESTIONS

1. Prove by induction that $(1+x)^n \geq 1+nx$ for $x>-1$ and for all $n \in \mathbb{N}$

2. Prove by induction that $\sum_{r=1}^{n} sinrx = \frac{sin\frac{1}{2}(n+1)xsin\frac{1}{2}nx}{sin\frac{1}{2}x}$ for all $n \in \mathbb{N}$

3. Prove by induction that $\prod_{r=1}^{n} r^2 = (n!)^2$ for all $n \in \mathbb{N}$

4. Prove by using the method of mathematical induction that $12^{n-1}+25^n$ is divisible by 13 for all $n \in \mathbb{N}$

5. Prove by induction that $12^n+8^{n+1}-2^{n+1}$ is divisible by 6 for all $n \in \mathbb{N}$

6. Prove by using the principle of mathematical induction that $\sum_{r=1}^{n} a+(r-1)d = \frac{n}{2}\left[2a+(n-1)d\right]$ where a and d are constants , and $n \in \mathbb{N}$

7. Given that x and y are positive , prove by induction that $(x+y)^n \geq x^n+y^n$ for all $n \in \mathbb{N}$

8. Prove by induction that $\frac{d^n(x^2e^{rx})}{dx^n} = r^nx^2e^{rx}+2nr^{n-1}xe^{rx}+n(n-1)r^{n-2}e^{rx}$ for all $n \in \mathbb{Z}^+$

9. Prove by induction that $\frac{d^n(xsinx)}{dx^n} = xsin(x+\frac{n}{2}\pi)+nsin(x+\frac{n-1}{2}\pi)$ for all $n \in \mathbb{Z}^+$

10. Prove the following statements by using the method of mathematical induction

    (a) 10^n-1 is divisible by 9 for all $n \in \mathbb{N}$

    (b) x^n-1 is divisible by $(x-1)$ for all $n \in \mathbb{N}$

11. (a) Show by using the principle of mathematical induction that $n^3+9n^2+26n+24$ is divisible by 6 for all $n \in \mathbb{N}$

    (b) Hence , prove that $n^4+10n^3+35n^2+50n+24$ is divisible by 24 for all $n \in \mathbb{N}$ using mathematical induction

12. Prove by induction that $n^2+3n+3^{n^2+n}$ is divisible by 2 for all $n \in \mathbb{N}$

13. Given $A = \begin{pmatrix} 2 & 9 \\ 1 & 2 \end{pmatrix}$, prove by induction that

$$A^n = \begin{pmatrix} \frac{1}{2}\cdot(-1)^n+\frac{1}{2}\cdot 5^n & \frac{3}{2}\cdot(-1)^{n+1}+\frac{3}{2}\cdot 5^n \\ \frac{1}{6}\cdot(-1)^{n+1}+\frac{1}{6}\cdot 5^n & \frac{1}{2}\cdot(-1)^n+\frac{1}{2}\cdot 5^n \end{pmatrix}$$

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